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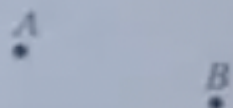
Loci 1

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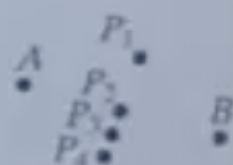
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A *locus* is a set of points that satisfies particular conditions.
The plural of *locus* is *loci*.

Given two random points, A and B , let's find the locus of point P , such that $PA = PB$.



As shown in the diagram, there are many points P , where $PA = PB$.



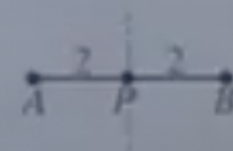
To find the set of P , we draw a line connecting the points.



$\therefore$ The locus of point P from points A and B is a line that is perpendicular to AB and divides AB in half.



- Given two fixed points, A and B , and a point P on AB whose distance from each point A and B is 2 units, draw several other points that also have an equal distance from points A and B .



$\therefore$ These points lie on a line.

N 1 b

2. Obtain the equation of the locus of point $P(x, y)$ such that the distance from P to each point $O(0, 0)$ and $B(5, 0)$ is the same.

$$[\text{Sol}] \quad PO = \sqrt{x^2 + y^2}$$

$$PB = \sqrt{(x-5)^2 + y^2}$$

$$\text{Letting } PO = PB,$$

$$\sqrt{x^2 + y^2} = \sqrt{(x-5)^2 + y^2}$$

$$x^2 + y^2 = (x-5)^2 + y^2$$

$$10x = 25$$

$$x = \frac{5}{2}$$

$$\therefore \text{The equation is: } x = \frac{5}{2}$$

3. Obtain the equation of the locus of point $P(x, y)$ such that the distance from P to each point $A(1, 2)$ and $B(-3, -2)$ is the same.

$$[\text{Sol}] \quad PA = \sqrt{(x-1)^2 + (y-2)^2}$$

$$PB = \sqrt{(x+3)^2 + (y+2)^2}$$

$$\text{Letting } PA = PB,$$

$$\sqrt{(x-1)^2 + (y-2)^2} = \sqrt{(x+3)^2 + (y+2)^2}$$

$$(x-1)^2 + (y-2)^2 = (x+3)^2 + (y+2)^2$$

$$8x + 8y = -8$$

$$x + y = -1$$

$$\therefore \text{The equation is: } x + y = -1$$

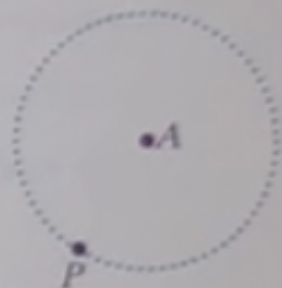
N 2 a

Loci 1

Time : to : Date Name

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1. Given a fixed point A and a point P whose distance from A is 2, draw several other points whose distance from A is also 2.



∴ These points lie on a circle with radius 2.

2. Obtain the equation of the locus of point $P(x, y)$ such that the distance from P to the origin is 3.

[Sol] Since the distance between (x, y) and $(0, 0)$ is 3,

$$\sqrt{x^2 + y^2} = 3$$

$$x^2 + y^2 = 9$$

∴ The equation is: $x^2 + y^2 = 9$

3. Obtain the equation of the locus of point $P(x, y)$ such that the distance from P to the origin is r .

[Sol] Since the distance between (x, y) and $(0, 0)$ is r ,

$$\sqrt{x^2 + y^2} = r$$

$$x^2 + y^2 = r^2$$

∴ The equation is: $x^2 + y^2 = r^2$

N 2 b

4. Obtain the equation of the locus of point $P(x, y)$ such that the distance from P to $A(5, 4)$ is 2.

[Sol] Since the distance between $P(x, y)$ and $A(5, 4)$ is 2,

$$\sqrt{(x-5)^2 + (y-4)^2} = 2$$

Squaring both sides,

$$(x-5)^2 + (y-4)^2 = 4$$

$\therefore$ The equation is: $(x-5)^2 + (y-4)^2 = 4$

5. Obtain the equation of the locus of point $P(x, y)$ such that the distance from P to each of the following points is 3, and then draw both graphs on the grid below.

- (1) $A(2, -1)$

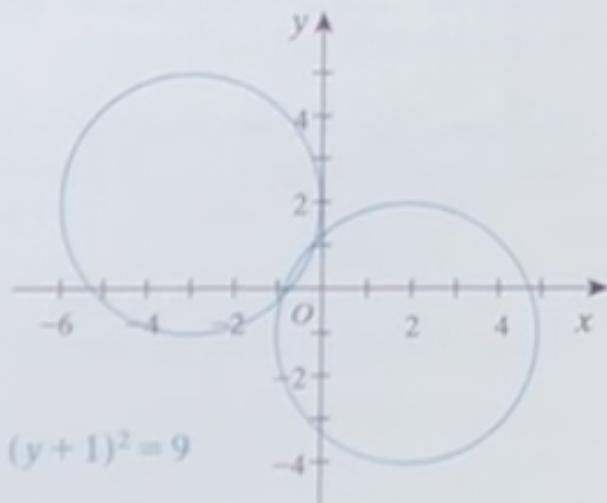
[Sol] Since the distance between $P(x, y)$ and $A(2, -1)$ is 3,

$$\sqrt{(x-2)^2 + (y+1)^2} = 3$$

Squaring both sides,

$$(x-2)^2 + (y+1)^2 = 9$$

$\therefore$ The equation is: $(x-2)^2 + (y+1)^2 = 9$



- (2) $B(-3, 2)$

[Sol] Since the distance between $P(x, y)$ and $A(-3, 2)$ is 3,

$$\sqrt{(x+3)^2 + (y-2)^2} = 3$$

Squaring both sides,

$$(x+3)^2 + (y-2)^2 = 9$$

$\therefore$ The equation is: $(x+3)^2 + (y-2)^2 = 9$

Time : to : Date Name

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1. Given $A(-3, 0)$, $B(3, 0)$ and $P(x, y)$, obtain the equation of the locus of point P , such that $AP^2 - BP^2 = 24$.

[Sol] Using the distance formula,

$$AP = \sqrt{(x+3)^2 + y^2} \quad \text{and}$$

$$BP = \sqrt{(x-3)^2 + y^2}$$

$$\begin{cases} AP^2 = (x+3)^2 + y^2 & \dots \textcircled{1} \\ BP^2 = (x-3)^2 + y^2 & \dots \textcircled{2} \end{cases}$$

Substituting $\textcircled{1}$ and $\textcircled{2}$ into $AP^2 - BP^2 = 24$,

$$(x+3)^2 + y^2 - [(x-3)^2 + y^2] = 24$$

$$12x = 24$$

$$\therefore x = 2$$

$\therefore$ The equation is: $x = 2$

2. Given the three points $A(-5, 0)$, $B(5, 0)$ and $C(3, 6)$, obtain the equation of the locus of point $P(x, y)$, such that $PA^2 + PB^2 + PC^2 = 170$.

$$[\text{Sol}] \begin{cases} PA^2 = (x+5)^2 + y^2 & \dots \textcircled{1} \\ PB^2 = (x-5)^2 + y^2 & \dots \textcircled{2} \\ PC^2 = (x-3)^2 + (y-6)^2 & \dots \textcircled{3} \end{cases}$$

Substituting $\textcircled{1}$, $\textcircled{2}$ and $\textcircled{3}$ into $PA^2 + PB^2 + PC^2 = 170$,

$$[(x+5)^2 + y^2] + [(x-5)^2 + y^2] + [(x-3)^2 + (y-6)^2] = 170$$

$$x^2 - 2x + y^2 - 4y - 25 = 0$$

$$(x^2 - 2x + 1) - 1 + (y^2 - 4y + 4) - 4 - 25 = 0$$

$$(x-1)^2 + (y-2)^2 = 30$$

$\therefore$ The equation is: $(x-1)^2 + (y-2)^2 = 30$

N 3 b

3. Given the two points $A(-1, 0)$ and $B(1, 0)$, obtain the equation of the locus of point $P(x, y)$, such that $PA^2 - PB^2 = 1$.

$$[\text{Sol}] \begin{cases} PA^2 = (x + 1)^2 + y^2 & \dots \textcircled{1} \\ PB^2 = (x - 1)^2 + y^2 & \dots \textcircled{2} \end{cases}$$

Substituting $\textcircled{1}$ and $\textcircled{2}$ into $PA^2 - PB^2 = 1$,

$$(x + 1)^2 + y^2 - (x - 1)^2 - y^2 = 1$$

$$4x = 1$$

$$x = \frac{1}{4}$$

$\therefore$ The equation is: $x = \frac{1}{4}$

4. Given the four points $A(0, 0)$, $B(-1, 0)$, $C(1, 0)$ and $D(4, 8)$, obtain the equation of the locus of point $P(x, y)$, such that $AP^2 + BP^2 + CP^2 + DP^2 = 78$.

$$[\text{Sol}] \begin{cases} AP^2 = x^2 + y^2 & \dots \textcircled{1} \\ BP^2 = (x + 1)^2 + y^2 & \dots \textcircled{2} \\ CP^2 = (x - 1)^2 + y^2 & \dots \textcircled{3} \\ DP^2 = (x - 4)^2 + (y - 8)^2 & \dots \textcircled{4} \end{cases}$$

Substituting $\textcircled{1}$, $\textcircled{2}$, $\textcircled{3}$ and $\textcircled{4}$ into $AP^2 + BP^2 + CP^2 + DP^2 = 78$ and simplifying,

$$x^2 - 2x + y^2 - 4y + 1 = 0,$$

$$(x - 1)^2 + (y - 2)^2 = 4$$

$\therefore$ The equation is: $(x - 1)^2 + (y - 2)^2 = 4$

Time : to : Date Name

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Ex.

Given points $A(-2, 0)$ and $B(2, 0)$, obtain the equation of the locus of point P , such that $AP : BP = 5 : 3$.

[Sol] Letting point P be at (x, y) ,

$$AP = \sqrt{(x+2)^2 + y^2} \text{ and } BP = \sqrt{(x-2)^2 + y^2}$$

$$\text{Since } AP : BP = 5 : 3, \quad 3(AP) = 5(BP)$$

$$3\sqrt{(x+2)^2 + y^2} = 5\sqrt{(x-2)^2 + y^2}$$

$$9[(x+2)^2 + y^2] = 25[(x-2)^2 + y^2]$$

$$2x^2 - 17x + 2y^2 + 8 = 0$$

$$x^2 - \frac{17}{2}x + y^2 = -4$$

$$\left(x - \frac{17}{4}\right)^2 + y^2 = \frac{225}{16}$$

$$\therefore \text{The equation is: } \left(x - \frac{17}{4}\right)^2 + y^2 = \frac{225}{16}$$

1. Given points $A(1, -2)$ and $B(6, 8)$, obtain the equation of the locus of point P , such that $AP : BP = 3 : 2$.

[Sol] Letting point P be at (x, y) ,

$$AP = \sqrt{(x-1)^2 + (y+2)^2} \text{ and } BP = \sqrt{(x-6)^2 + (y-8)^2}$$

$$\text{Since } AP : BP = 3 : 2, \quad 2(AP) = 3(BP)$$

$$2\sqrt{(x-1)^2 + (y+2)^2} = 3\sqrt{(x-6)^2 + (y-8)^2}$$

$$4[(x-1)^2 + (y+2)^2] = 9[(x-6)^2 + (y-8)^2]$$

$$x^2 - 20x + y^2 - 32y + 176 = 0$$

$$(x-10)^2 + (y-16)^2 = 180$$

$$\therefore \text{The equation is: } (x-10)^2 + (y-16)^2 = 180$$

N 4 b

2. Given points $A(1, 0)$ and $B(0, 1)$, obtain the equation of the locus of point P , such that $AP : BP = 2 : 3$.

[Sol] Letting point P be at (x, y) ,

$$AP = \sqrt{(x-1)^2 + y^2} \quad \text{and} \quad BP = \sqrt{x^2 + (y-1)^2}$$

$$\text{Since } AP : BP = 2 : 3, \quad 3(AP) = 2(BP)$$

$$3\sqrt{(x-1)^2 + y^2} = 2\sqrt{x^2 + (y-1)^2}$$

$$9[(x-1)^2 + y^2] = 4[x^2 + (y-1)^2]$$

$$5x^2 - 18x + 5y^2 + 8y + 5 = 0$$

$$\left(x - \frac{9}{5}\right)^2 + \left(y + \frac{4}{5}\right)^2 = \frac{72}{25}$$

$$\therefore \text{The equation is: } \left(x - \frac{9}{5}\right)^2 + \left(y + \frac{4}{5}\right)^2 = \frac{72}{25}$$

3. Given points $A(2, -1)$ and $B(3, 2)$, obtain the equation of the locus of point P , such that $AP : BP = 1 : 2$.

[Sol] Letting point P be at (x, y) ,

$$AP = \sqrt{(x-2)^2 + (y+1)^2} \quad \text{and} \quad BP = \sqrt{(x-3)^2 + (y-2)^2}$$

$$\text{Since } AP : BP = 1 : 2, \quad 2(AP) = BP$$

$$2\sqrt{(x-2)^2 + (y+1)^2} = \sqrt{(x-3)^2 + (y-2)^2}$$

$$4[(x-2)^2 + (y+1)^2] = (x-3)^2 + (y-2)^2$$

$$3x^2 - 10x + 3y^2 + 12y + 7 = 0$$

$$\left(x - \frac{5}{3}\right)^2 + (y+2)^2 = \frac{40}{9}$$

$$\therefore \text{The equation is: } \left(x - \frac{5}{3}\right)^2 + (y+2)^2 = \frac{40}{9}$$

Time : to : Date Name

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1. Given the fixed points $F(3, 0)$ and $F'(-3, 0)$, obtain the equation of the locus of point $P(x, y)$, such that $PF + PF' = 10$.

[Sol] Since $PF + PF' = 10$,

$$\sqrt{(x-3)^2 + y^2} + \sqrt{(x+3)^2 + y^2} = 10$$

$$\sqrt{(x-3)^2 + y^2} = 10 - \sqrt{(x+3)^2 + y^2}$$

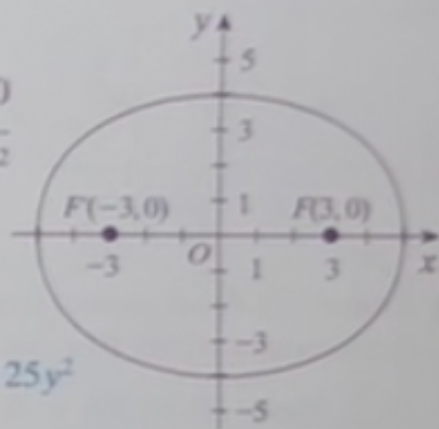
Squaring both sides and simplifying,

$$3x + 25 = 5\sqrt{(x+3)^2 + y^2}$$

$$9x^2 + 150x + 625 = 25x^2 + 150x + 225 + 25y^2$$

$$16x^2 + 25y^2 = 400$$

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$



2. Given the fixed points $F(0, 2)$ and $F'(0, -2)$, obtain the equation of the locus of point $P(x, y)$, such that $PF + PF' = 8$, and then draw the graph of the locus.

[Sol] Since $PF + PF' = 8$,

$$\sqrt{x^2 + (y-2)^2} + \sqrt{x^2 + (y+2)^2} = 8$$

$$\sqrt{x^2 + (y-2)^2} = 8 - \sqrt{x^2 + (y+2)^2}$$

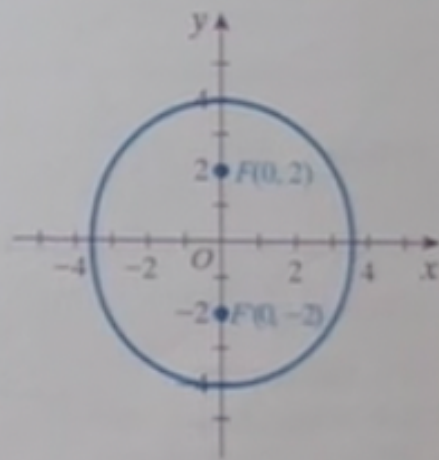
Squaring both sides and simplifying,

$$y + 8 = 2\sqrt{x^2 + (y+2)^2}$$

$$y^2 + 16y + 64 = 4x^2 + 4y^2 + 16y + 16$$

$$4x^2 + 3y^2 = 48$$

$$\frac{x^2}{12} + \frac{y^2}{16} = 1$$



This type of locus in the preceding two problems is called an *ellipse*, with F and F' as its *foci*.

N 5 b

3. Given the fixed points $F(\sqrt{5}, 0)$ and $F'(-\sqrt{5}, 0)$, obtain the equation of the locus of point $P(x, y)$, such that $PF + PF' = 6$.

[Sol] Since $PF + PF' = 6$,

$$\sqrt{(x - \sqrt{5})^2 + y^2} + \sqrt{(x + \sqrt{5})^2 + y^2} = 6$$

$$\sqrt{(x - \sqrt{5})^2 + y^2} = 6 - \sqrt{(x + \sqrt{5})^2 + y^2}$$

$$x\sqrt{5} + 9 = 3\sqrt{(x + \sqrt{5})^2 + y^2}$$

$$4x^2 + 9y^2 = 36$$

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

4. Given the fixed points $F(0, \sqrt{5})$ and $F'(0, -\sqrt{5})$, obtain the equation of the locus of point $P(x, y)$, such that $PF + PF' = 6$.

[Sol] Since $PF + PF' = 6$,

$$\sqrt{x^2 + (y - \sqrt{5})^2} + \sqrt{x^2 + (y + \sqrt{5})^2} = 6$$

$$\sqrt{x^2 + (y - \sqrt{5})^2} = 6 - \sqrt{x^2 + (y + \sqrt{5})^2}$$

$$y\sqrt{5} + 9 = 3\sqrt{x^2 + (y + \sqrt{5})^2}$$

$$9x^2 + 4y^2 = 36$$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

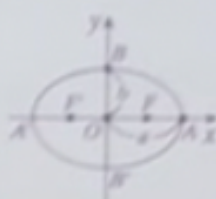
An equation of this form is called the *standard form of an ellipse*.

- When $a^2 > b^2$,

O is the center.

AA' is the major axis.

BB' is the minor axis.



The foci F and F' are:

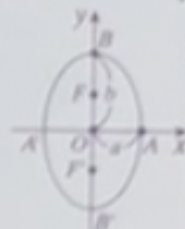
$$(\pm\sqrt{a^2 - b^2}, 0)$$

- When $a^2 < b^2$,

O is the center.

BB' is the major axis.

AA' is the minor axis.



The foci F and F' are:

$$(0, \pm\sqrt{b^2 - a^2})$$

N 6 a

Loc 1

Time : to : Date Name

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1. Given the fixed points $F(5, 0)$ and $F'(-5, 0)$, obtain the equation of the locus of point $P(x, y)$, such that $PF - PF' = 8$.

[Sol] Since $PF - PF' = 8$,

$$\sqrt{(x-5)^2 + y^2} - \sqrt{(x+5)^2 + y^2} = 8$$

$$\sqrt{(x-5)^2 + y^2} = 8 + \sqrt{(x+5)^2 + y^2}$$

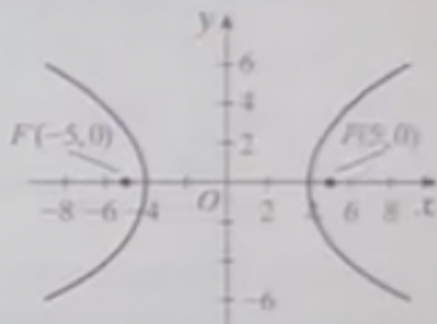
Squaring both sides and simplifying,

$$-16 - 5x = 4\sqrt{(x+5)^2 + y^2}$$

$$256 + 160x + 25x^2 = 16x^2 + 160x + 400 + 16y^2$$

$$9x^2 - 16y^2 = 144$$

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$



2. Given the fixed points $F(4, 0)$ and $F'(-4, 0)$, obtain the equation of the locus of point $P(x, y)$, such that $PF - PF' = 6$, and then draw the graph of the locus.

[Sol] Since $PF - PF' = 6$,

$$\sqrt{(x-4)^2 + y^2} - \sqrt{(x+4)^2 + y^2} = 6$$

$$\sqrt{(x-4)^2 + y^2} = 6 + \sqrt{(x+4)^2 + y^2}$$

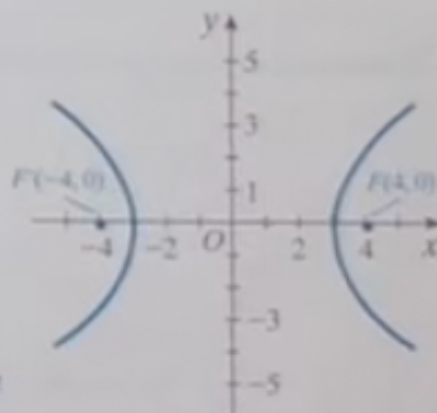
Squaring both sides and simplifying,

$$-4x - 9 = 3\sqrt{(x+4)^2 + y^2}$$

$$16x^2 + 72x + 81 = 9x^2 + 72x + 144 + 9y^2$$

$$7x^2 - 9y^2 = 63$$

$$\frac{x^2}{9} - \frac{y^2}{7} = 1$$



This type of locus in the preceding two problems is called a *hyperbola*, with F and F' as its *foci*.

N 6 b

3. Given the fixed points $F(0, 6)$ and $F'(0, -6)$, obtain the equation of the locus of point $P(x, y)$, such that $PF - PF' = -10$.

[Sol] Since $PF - PF' = -10$,

$$\sqrt{x^2 + (y-6)^2} - \sqrt{x^2 + (y+6)^2} = -10$$

$$\sqrt{x^2 + (y-6)^2} = -10 + \sqrt{x^2 + (y+6)^2}$$

Squaring both sides and simplifying,

$$5\sqrt{x^2 + (y+6)^2} = 25 + 6y$$

$$25x^2 - 11y^2 = -275$$

$$\frac{x^2}{11} - \frac{y^2}{25} = -1$$

4. Given the fixed points $F(-6, 0)$ and $F'(6, 0)$, obtain the equation of the locus of point $P(x, y)$, such that $PF - PF' = 10$.

[Sol] Since $PF - PF' = 10$,

$$\sqrt{(x+6)^2 + y^2} - \sqrt{(x-6)^2 + y^2} = 10$$

$$\sqrt{(x+6)^2 + y^2} = 10 + \sqrt{(x-6)^2 + y^2}$$

Squaring both sides and simplifying,

$$5\sqrt{(x-6)^2 + y^2} = 6x - 25$$

$$11x^2 - 25y^2 = 275$$

$$\frac{x^2}{25} - \frac{y^2}{11} = 1$$

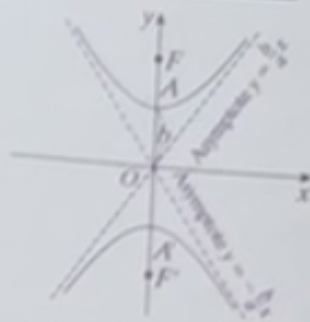
Hyperbolas have the following two standard forms:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



The foci F and F' are:
 $(\pm\sqrt{a^2 + b^2}, 0)$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$



The foci F and F' are:
 $(0, \pm\sqrt{a^2 + b^2})$

Time : to : Date Name

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1. Given the fixed point $F(1, 0)$ and the line $x = 2$, obtain the equation of the locus of point $P(x, y)$, such that P is equidistant from F and the line.

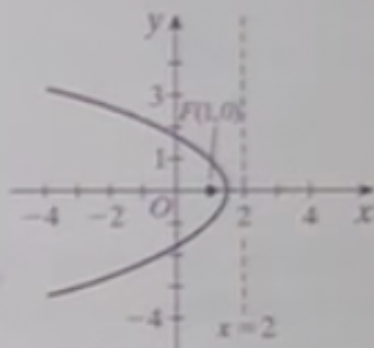
[Sol] $\sqrt{(x-1)^2 + y^2} = |x - 2|$

Squaring both sides,

$$(x-1)^2 + y^2 = (x-2)^2$$

$$2x + y^2 = 3$$

$$y^2 = -2x + 3$$



∴ The equation is: $y^2 = -2x + 3$

2. Given the fixed point $F\left(-\frac{1}{2}, 0\right)$ and the line $x = \frac{1}{2}$, obtain the equation of the locus of point $P(x, y)$, such that P is equidistant from F and the line.

[Sol] $\sqrt{\left(x + \frac{1}{2}\right)^2 + y^2} = \left|x - \frac{1}{2}\right|$

Squaring both sides,

$$\left(x + \frac{1}{2}\right)^2 + y^2 = \left(x - \frac{1}{2}\right)^2$$

$$2x + y^2 = 0$$

$$y^2 = -2x$$

∴ The equation is: $y^2 = -2x$

This type of locus in the preceding two problems is called a *parabola*, with F as its *focus*.

N 7 b

3. Given the fixed point $F(3, 0)$ and the line $x = -3$, obtain the equation of the locus of point $P(x, y)$, such that P is equidistant from F and the line.

[Sol] $\sqrt{(x-3)^2 + y^2} = |x+3|$

Squaring both sides,

$$(x-3)^2 + y^2 = (x+3)^2$$

$$-12x + y^2 = 0$$

$$y^2 = 12x$$

$\therefore$ The equation is: $y^2 = 12x$

4. Given the fixed point $F(0, 3)$ and the line $y = -3$, obtain the equation of the locus of point $P(x, y)$, such that P is equidistant from F and the line.

[Sol] $\sqrt{x^2 + (y-3)^2} = |y+3|$

Squaring both sides,

$$x^2 + (y-3)^2 = (y+3)^2$$

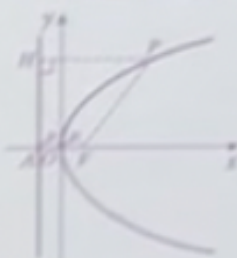
$$x^2 - 12y = 0$$

$$x^2 = 12y$$

$\therefore$ The equation is: $x^2 = 12y$

Parabolas have the following two standard forms (when $p > 0$):

$$y^2 = 4px$$



O is the vertex.

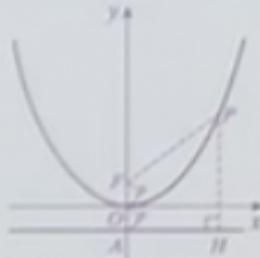
The x -axis is the axis of symmetry.

Line $x = -p$ is called the *directrix*.

Point F is the focus.

$$PH = PF, OF = OA = p$$

$$x^2 = 4py$$



O is the vertex.

The y -axis is the axis of symmetry.

Line $y = -p$ is called the *directrix*.

Point F is the focus.

$$PH = PF, OF = OA = p$$

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	-	-	-

1. Given a point $P(x, y)$ such that the ratio of the distances from P to fixed point $F(2, 0)$, and from P to line $x = -1$ is $2 : 1$, obtain the equation of the locus of point P .

[Sol] Letting the point of intersection of the perpendicular dropped from P and the line $x = -1$ be $H(-1, y)$,

$$PF : PH = 2 : 1$$

$$PF = \sqrt{(x-2)^2 + y^2}$$

$$PH = \sqrt{(x+1)^2} = |x+1|$$

$$\therefore \sqrt{(x-2)^2 + y^2} = 2|x+1|$$

Squaring both sides and simplifying,

$$3(x+2)^2 - y^2 = 12$$

$$\therefore \text{The equation is: } \frac{(x+2)^2}{4} - \frac{y^2}{12} = 1$$

$$[3x^2 + 12x - y^2 = 0]$$

2. Given a point $P(x, y)$ such that the ratio of the distances from P to fixed point $F(0, -1)$, and from P to line $y = 2$ is $1 : 1$, obtain the equation of the locus of point P .

[Sol] Letting the point of intersection of the perpendicular dropped from P and the line $y = 2$ be $H(x, 2)$,

$$PF : PH = 1 : 1$$

$$PF = \sqrt{x^2 + (y+1)^2}$$

$$PH = \sqrt{(y-2)^2} = |y-2|$$

$$\therefore \sqrt{x^2 + (y+1)^2} = |y-2|$$

Squaring both sides and simplifying,

$$x^2 + 6y = 3$$

$$\therefore \text{The equation is: } x^2 = -6y + 3$$

N 8 b

3. Given a point $P(x, y)$, such that the ratio of the distances from P to fixed points $F(3, 0)$ and $F'(-3, 0)$ is $1 : 2$, obtain the equation of the locus of point P .

$$\begin{aligned} \text{[Sol]} \quad PF &= \sqrt{(x-3)^2 + y^2} \\ PF' &= \sqrt{(x+3)^2 + y^2} \end{aligned}$$

$$\text{Since } PF : PF' = 1 : 2,$$

$$2\sqrt{(x-3)^2 + y^2} = \sqrt{(x+3)^2 + y^2}$$

Squaring both sides,

$$4[(x-3)^2 + y^2] = (x+3)^2 + y^2$$

$$x^2 - 10x + y^2 = -9$$

$$x^2 - 10x + 25 + y^2 = 16$$

$$(x-5)^2 + y^2 = 16$$

$$\therefore \text{The equation is: } (x-5)^2 + y^2 = 16$$

Time : to : Date Name

100%	90%	80%	70%	60%~
(mistakes) 0	-	-	-	1-

1. Given that line segment AB has length 6 with moving endpoint A on the x -axis and moving endpoint B on the y -axis, obtain the equation of the locus of point P which divides AB internally in the ratio of 1 : 2.

[Sol] Placing A , B , and P at $(a, 0)$, $(0, b)$ and (X, Y) .

$$X = \frac{2}{3}a, \quad Y = \frac{b}{3}$$

$$a = \frac{3}{2}X, \quad b = 3Y \quad \dots \textcircled{1}$$

Since AB has length 6,

$$a^2 + b^2 = 36 \quad \dots \textcircled{2}$$

Substituting $\textcircled{1}$ into $\textcircled{2}$,

$$\left(\frac{3}{2}X\right)^2 + (3Y)^2 = 36$$

$$\frac{X^2}{16} + \frac{Y^2}{4} = 1$$

$$\therefore \text{The equation is: } \frac{X^2}{16} + \frac{Y^2}{4} = 1$$

2. Given that line segment AB has length 2 with moving endpoint A on the y -axis and moving endpoint B on the x -axis, obtain the equation of the locus of point P which divides AB internally in the ratio of 4 : 3.

[Sol] Placing A , B , and P at $(0, a)$, $(b, 0)$ and (X, Y) .

$$X = \frac{4}{7}b, \quad Y = \frac{3}{7}a$$

$$b = \frac{7}{4}X, \quad a = \frac{7}{3}Y \quad \dots \textcircled{1}$$

Since AB has length 2,

$$a^2 + b^2 = 4 \quad \dots \textcircled{2}$$

Substituting $\textcircled{1}$ into $\textcircled{2}$,

$$\left(\frac{7}{3}Y\right)^2 + \left(\frac{7}{4}X\right)^2 = 4$$

$$\frac{49}{16}X^2 + \frac{49}{9}Y^2 = 4$$

$$\therefore \text{The equation is: } \frac{49}{64}X^2 + \frac{49}{36}Y^2 = 1$$

N 9 b

3. Given that the equation of the locus of point $P(x, y)$ is $\frac{x^2}{9} + \frac{y^2}{4} = 1$, and that its foci are $F(\sqrt{5}, 0)$ and $F'(-\sqrt{5}, 0)$, determine the value of $PF + PF'$.

[Sol] Letting $PF + PF' = c$,

$$\begin{aligned}\sqrt{(x - \sqrt{5})^2 + y^2} + \sqrt{(x + \sqrt{5})^2 + y^2} &= c \\ \sqrt{(x - \sqrt{5})^2 + y^2} &= c - \sqrt{(x + \sqrt{5})^2 + y^2} \\ c^2 + 4\sqrt{5}x &= 2c\sqrt{(x + \sqrt{5})^2 + y^2} \\ c^4 + 8\sqrt{5}c^2x + 80x^2 &= 4c^2[(x + \sqrt{5})^2 + y^2] \\ 4c^2x^2 - 80x^2 + 4c^2y^2 &= c^4 - 20c^2 \\ 4x^2(c^2 - 20) + 4c^2y^2 &= c^2(c^2 - 20) \\ \frac{4x^2}{c^2} + \frac{4y^2}{c^2 - 20} &= 1\end{aligned}$$

Since the equation of the locus is $\frac{x^2}{9} + \frac{y^2}{4} = 1$,

$$\begin{aligned}\frac{4}{c^2} &= \frac{1}{9} \\ \therefore c &= 6 \quad (\because PF + PF' \geq 0)\end{aligned}$$

Thus, $PF + PF' = 6$

Time : to : Date Name

100%	90%	80%	70%
(mistakes) 0	-	-	1

1. Given points $A(-4, 0)$, $B(4, 0)$ and $P(x, y)$, obtain the equation of the locus of point P , such that $AP^2 - BP^2 = 24$.

[Sol] Using the distance formula,

$$AP = \sqrt{(x+4)^2 + y^2} \text{ and}$$

$$BP = \sqrt{(x-4)^2 + y^2}$$

$$\begin{cases} AP^2 = (x+4)^2 + y^2 & \dots \textcircled{1} \\ BP^2 = (x-4)^2 + y^2 & \dots \textcircled{2} \end{cases}$$

Substituting $\textcircled{1}$ and $\textcircled{2}$ into $AP^2 - BP^2 = 24$,

$$(x+4)^2 + y^2 - [(x-4)^2 + y^2] = 24$$

$$16x = 24$$

$$x = \frac{3}{2}$$

$\therefore$ The equation is: $x = \frac{3}{2}$

2. Given the fixed points $F(0, \sqrt{5})$ and $F'(0, -\sqrt{5})$, obtain the equation of the locus of point $P(x, y)$, such that $PF + PF' = 8$.

[Sol] Since $PF + PF' = 8$,

$$\sqrt{x^2 + (y - \sqrt{5})^2} + \sqrt{x^2 + (y + \sqrt{5})^2} = 8$$

$$\sqrt{x^2 + (y - \sqrt{5})^2} = 8 - \sqrt{x^2 + (y + \sqrt{5})^2}$$

$$16 + y\sqrt{5} = 4\sqrt{x^2 + (y + \sqrt{5})^2}$$

$$16x^2 + 11y^2 = 176$$

$$\frac{x^2}{11} + \frac{y^2}{16} = 1$$

$\therefore$ The equation is: $\frac{x^2}{11} + \frac{y^2}{16} = 1$

N 10 b

3. Given the fixed point $F(4, 0)$ and the line $x = -4$, obtain the equation of the locus of point $P(x, y)$, such that P is equidistant from F and the line.

[Sol] $\sqrt{(x-4)^2 + y^2} = x + 4$

Squaring both sides,

$$(x-4)^2 + y^2 = (x+4)^2$$

$$-16x + y^2 = 0$$

$$y^2 = 16x$$

4. Given that line segment AB has length 6 with moving endpoint A on the x -axis and moving endpoint B on the y -axis, obtain the equation of the locus of point P which divides AB internally in the ratio of 2 : 1.

[Sol] Placing A , B , and P at $(a, 0)$, $(0, b)$ and (X, Y) ,

$$X = \frac{a}{3}, \quad Y = \frac{2}{3}b$$

$$a = 3X, \quad b = \frac{3}{2}Y \quad \dots \textcircled{1}$$

Since AB has length 6,

$$a^2 + b^2 = 36 \quad \dots \textcircled{2}$$

Substituting $\textcircled{1}$ into $\textcircled{2}$,

$$(3X)^2 + \left(\frac{3}{2}Y\right)^2 = 36$$

$$\frac{X^2}{4} + \frac{Y^2}{16} = 1$$

$$\therefore \text{The equation is: } \frac{X^2}{4} + \frac{Y^2}{16} = 1$$

Loc 2

Time : to : Date Name

100%	90%	80%	70%	60%~
(mistakes) 0	-	-	1	2~

1. Obtain the equation of the locus of the midpoint, M , of the line segment connecting the fixed point $A(2, 3)$ and a moving point P on the line $3x - 5y - 2 = 0$.

[Sol] Let (X, Y) and (x, y) be the coordinates of M and P , respectively.

$$\begin{cases} X = \frac{x+2}{2} & \dots \textcircled{1} \\ Y = \frac{y+3}{2} & \dots \textcircled{2} \\ 3x - 5y - 2 = 0 & \dots \textcircled{3} \end{cases}$$

Using $\textcircled{1}$ and $\textcircled{2}$,

$$x = 2X - 2 \quad \dots \textcircled{4}$$

$$y = 2Y - 3 \quad \dots \textcircled{5}$$

Substituting $\textcircled{4}$ and $\textcircled{5}$ into $\textcircled{3}$,

$$6X - 10Y + 7 = 0$$

$\therefore$ The equation of the locus is $6X - 10Y + 7 = 0$.

2. Given a line segment AB of constant length $2l$ with moving endpoint A on the x -axis and moving endpoint B on the y -axis, obtain the equation of the locus of the midpoint P of AB .

[Sol] Let $(a, 0)$, $(0, b)$ and (X, Y) be the coordinates of A , B and P , respectively.

$$\begin{cases} X = \frac{a}{2} & \dots \textcircled{1} \\ Y = \frac{b}{2} & \dots \textcircled{2} \\ a^2 + b^2 = 4l^2 & \dots \textcircled{3} \end{cases}$$

Using $\textcircled{1}$ and $\textcircled{2}$,

$$a = 2X \quad \dots \textcircled{4}$$

$$b = 2Y \quad \dots \textcircled{5}$$

Substituting $\textcircled{4}$ and $\textcircled{5}$ into $\textcircled{3}$,

$$X^2 + Y^2 = l^2$$

$\therefore$ The equation of the locus is the circle with center $(0, 0)$ and radius l .

N 11 b

3. Obtain the equation of the locus of the midpoint M of the line segment connecting the fixed point $A(3, 1)$ and a moving point P on the parabola $y = x^2$.

[Sol] Let (X, Y) and (x, y) be the coordinates of M and P , respectively.

$$\begin{cases} X = \frac{x+3}{2} & \dots \textcircled{1} \\ Y = \frac{y+1}{2} & \dots \textcircled{2} \\ y = x^2 & \dots \textcircled{3} \end{cases}$$

Using $\textcircled{1}$ and $\textcircled{2}$,

$$x = 2X - 3 \quad \dots \textcircled{4}$$

$$y = 2Y - 1 \quad \dots \textcircled{5}$$

Substituting $\textcircled{4}$ and $\textcircled{5}$ into $\textcircled{3}$, $Y = 2X^2 - 6X + 5$

$\therefore$ The equation of the locus is the parabola $Y = 2X^2 - 6X + 5$.

4. Obtain the equation of the locus of the midpoint M of the line segment connecting the fixed point $A(6, 4)$ and a moving point P on the circle $x^2 + y^2 = 9$.

[Sol] Let (X, Y) and (x, y) be the coordinates of M and P , respectively.

$$\begin{cases} X = \frac{x+6}{2} & \dots \textcircled{1} \\ Y = \frac{y+4}{2} & \dots \textcircled{2} \\ x^2 + y^2 = 9 & \dots \textcircled{3} \end{cases}$$

Using $\textcircled{1}$ and $\textcircled{2}$,

$$x = 2X - 6 \quad \dots \textcircled{4}$$

$$y = 2Y - 4 \quad \dots \textcircled{5}$$

Substituting $\textcircled{4}$ and $\textcircled{5}$ into $\textcircled{3}$, $(X-3)^2 + (Y-2)^2 = \frac{9}{4}$

$\therefore$ The equation of the locus is the circle with center $(3, 2)$ and radius $\frac{3}{2}$.

Loci 2

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	-	-	1-

1. Dropping a perpendicular from point Q on circle $x^2 + y^2 = 25$ to the x -axis at point H , and placing point P on QH such that $PH = \frac{3}{5}QH$, obtain the equation of the locus of point P , and then graph it.

[Sol] Let (x, y) be the coordinates of point Q on $x^2 + y^2 = 25$... ①
and let (X, Y) be the coordinates of P .

$$X = x, \quad Y = \frac{3}{5}y$$

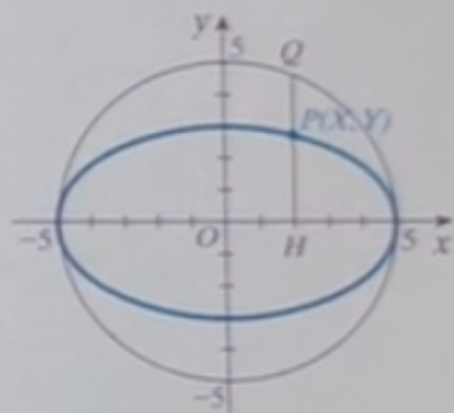
$$\therefore x = X, \quad y = \frac{5}{3}Y$$

Substituting these into ①,

$$X^2 + \left(\frac{5}{3}Y\right)^2 = 25$$

$$\frac{X^2}{25} + \frac{Y^2}{9} = 1$$

$\therefore$ The equation of the locus of point P is the ellipse $\frac{X^2}{25} + \frac{Y^2}{9} = 1$.



2. Dropping a perpendicular from point Q on circle $x^2 + y^2 = 9$ to the x -axis at point H , and placing point P on QH such that P divides QH internally in the ratio of 1:2, obtain the equation of the locus of point P , and then graph it.

[Sol] Let (x, y) be the coordinates of point Q on $x^2 + y^2 = 9$... ①
and let (X, Y) be the coordinates of P .

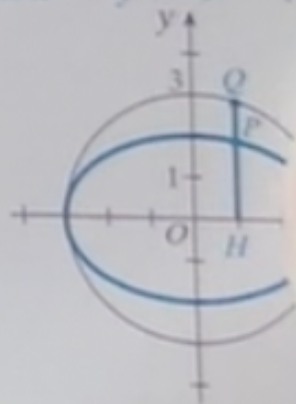
$$X = x, \quad Y = \frac{2}{3}y$$

$$\therefore x = X, \quad y = \frac{3}{2}Y$$

Substituting these into ①,

$$\frac{X^2}{9} + \frac{Y^2}{4} = 1$$

$\therefore$ The equation of the locus of point P is the ellipse $\frac{X^2}{9} + \frac{Y^2}{4} = 1$.



N 12 b

3. Given:

The circle $x^2 + y^2 - 8y - 9 = 0$... ① which passes through the two fixed points $A(-3, 0)$ and $B(3, 0)$, and $\triangle PAB$ whose vertex is an arbitrary moving point P on the circumference of ①.

Obtain: The equation of the locus of the center of gravity G of $\triangle PAB$.

[Sol] Let (X, Y) and (x, y) be the coordinates of G and P , respectively.

Since point $G(X, Y)$ divides the line segment connecting the origin O and $P(x, y)$ internally in the ratio of $1 : 2$,

$$X = \frac{x}{3}, \quad Y = \frac{y}{3}$$

$$\therefore x = 3X, \quad y = 3Y$$

Substituting these into ①,

$$(3X)^2 + (3Y)^2 - 8(3Y) - 9 = 0$$

$$9X^2 + 9Y^2 - 24Y - 9 = 0$$

$$\therefore X^2 + \left(Y - \frac{4}{3}\right)^2 = \frac{25}{9}$$

Therefore, the equation of the locus of point G is the circle with center $\left(0, \frac{4}{3}\right)$ and radius $\frac{5}{3}$, excluding points $(\pm 1, 0)$.

(Point P cannot coincide with A or B .)

Time : to : Date Name

100%	90%	80%	70%
(mistakes) 0	-	-	-

1. Obtain the equation of the locus of the points of intersection of each of the following pairs of lines for all real values of t .

$$(1) \begin{cases} x + y = 3t & \dots \textcircled{1} \\ x - y = -t + 2 & \dots \textcircled{2} \end{cases}$$

[Sol] From $\textcircled{2}$,

$$t = 2 - x + y$$

Substituting this value into $\textcircled{1}$,

$$x + y = 3(2 - x + y)$$

$$x + y = 6 - 3x + 3y$$

$$2x - y = 3$$

$\therefore$ The equation of the locus of the points of intersection is:

$$2x - y = 3$$

$$(2) \begin{cases} 3x - y = 4t - 1 & \dots \textcircled{1} \\ 5x + 2y = 3t + 2 & \dots \textcircled{2} \end{cases}$$

[Sol] From $\textcircled{1}$,

$$t = \frac{3x - y + 1}{4}$$

Substituting this value into $\textcircled{2}$,

$$5x + 2y = 3\left(\frac{3x - y + 1}{4}\right) + 2$$

$$20x + 8y = 9x - 3y + 3 + 8$$

$$x + y = 1$$

$\therefore$ The equation of the locus of the points of intersection is:

$$x + y = 1$$

N 13 b

Ex. Obtain the equation of the locus of the points of intersection of the two given lines.

$$\begin{cases} mx - y = m & \dots \textcircled{1} \\ x + my = -1 & \dots \textcircled{2} \end{cases}$$

[Sol] Rewriting $\textcircled{1}$,

$$m(x - 1) = y$$

(a) When $x \neq 1$,

$$m = \frac{y}{x - 1}$$

Substituting this into $\textcircled{2}$,

$$x^2 + y^2 = 1, \quad x \neq 1$$

(b) When $x = 1$,

$\textcircled{1}$ gives $y = 0$,

But $(x, y) = (1, 0)$ does not satisfy $\textcircled{2}$.

$\therefore$ The equation of the locus obtained from (a) and (b) is the circle $x^2 + y^2 = 1$, excluding the point $(1, 0)$.

2. Obtain the equation of the locus of the points of intersection of the two given lines.

$$\begin{cases} y + k(x - 2) = 0 & \dots \textcircled{1} \\ ky - (x + 2) = 0 & \dots \textcircled{2} \end{cases}$$

[Sol] Rewriting $\textcircled{1}$,

$$k(x - 2) = -y$$

(a) When $x \neq 2$,

$$k = -\frac{y}{x - 2}$$

Substituting this into $\textcircled{2}$,

$$x^2 + y^2 = 4, \quad x \neq 2$$

(b) When $x = 2$,

$\textcircled{1}$ gives $y = 0$,

But $(x, y) = (2, 0)$ does not satisfy $\textcircled{2}$.

$\therefore$ The equation of the locus obtained from (a) and (b) is the circle $x^2 + y^2 = 4$, excluding the point $(2, 0)$.

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	-	1	2-

1. Given the two lines: $mx - y = 0 \dots \textcircled{1}$ and $x + my - 1 = 0 \dots \textcircled{2}$:

- (1) Obtain the equation of the locus of their points of intersection for all real values of m .

[Sol] $\textcircled{1}$ gives $mx = y \dots \textcircled{3}$

(a) When $x \neq 0$, $\textcircled{3}$ gives $m = \frac{y}{x}$.

Substituting this into $\textcircled{2}$ and rearranging,

$$\left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}, \quad x \neq 0$$

(b) When $x = 0$, $\textcircled{3}$ gives $y = 0$.

But these values do not satisfy $\textcircled{2}$.

Thus, point $(0, 0)$ is not in the locus.

Therefore, the equation of the locus is the circle with center $\left(\frac{1}{2}, 0\right)$ and radius $\frac{1}{2}$, excluding the point $(0, 0)$.

- (2) Draw the path traced out by the points of intersection of the two lines when $0 \leq m \leq 1$. (Consider the various slopes of $\textcircled{1}$.)

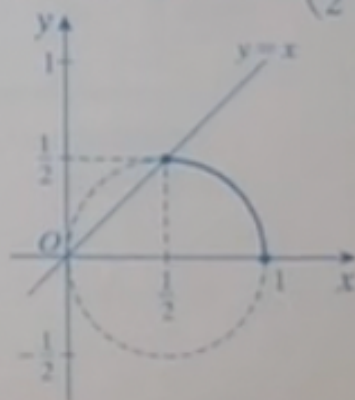
[Sol] $\textcircled{1}$ is a line with slope m passing through the origin.

When $m = 1$, solving $y = x$ and $\left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}$,

The point of intersection with the circle obtained in (1) is: $\left(\frac{1}{2}, \frac{1}{2}\right)$

When $m = 0$, the line becomes $y = 0$, and its point of intersection with the circle is $(1, 0)$.

Thus, when $0 \leq m \leq 1$ the locus is the solid part of the circle in the diagram.



N 14 b

2. Given the two lines: $mx - y + 2m = 0 \dots \textcircled{1}$ and $x + my + 6 = 0 \dots \textcircled{2}$:

- (1) Obtain the equation of the locus of their points of intersection for all real values of m .

[Sol] $\textcircled{1}$ gives $m(x + 2) = y \dots \textcircled{3}$

(a) When $x \neq -2$, $\textcircled{3}$ gives $m = \frac{y}{x+2}$.

Substituting this into $\textcircled{2}$ and rearranging,

$$(x + 4)^2 + y^2 = 4, \quad x \neq -2$$

(b) When $x = -2$, $\textcircled{3}$ gives $y = 0$.

But these values do not satisfy $\textcircled{2}$.

Thus, point $(-2, 0)$ is not in the locus.

Therefore, the equation of the locus is the circle with center $(-4, 0)$ and radius 2, excluding the point $(-2, 0)$.

- (2) Draw the path traced out by the points of intersection of the two lines when $0 \leq m \leq 1$.

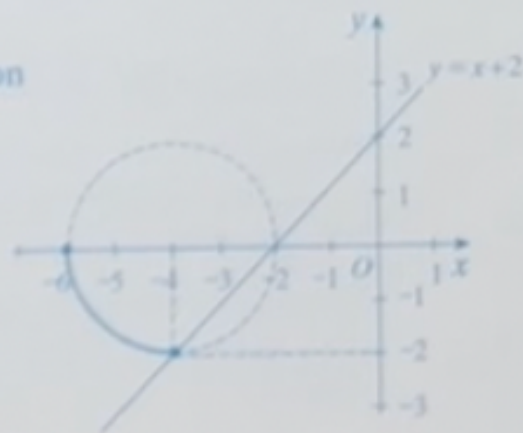
[Sol] $\textcircled{1}$ is a line with slope m passing through the point $(-2, 0)$.

When $m = 1$, solving $y = x + 2$ and $(x + 4)^2 + y^2 = 4$,

The point of intersection with the circle obtained in (1) is: $(-4, -2)$

When $m = 0$, the line becomes $y = 0$, and its point of intersection with the circle is $(-6, 0)$.

Thus, when $0 \leq m \leq 1$ the locus is the solid part of the circle in the diagram.



Time : to : Date Name

100%	90%	80%	70%
(mistakes) 0	-	-	1

1. Obtain the equation of the locus of the point $P(x, y)$ such that the lengths of the tangent lines from P to each of the following pairs of circles is equal.

$$(1) \begin{cases} x^2 + y^2 = 3 \\ (x+2)^2 + (y-2)^2 = 2 \end{cases}$$

[Sol] Since the tangent lines from P are equal,

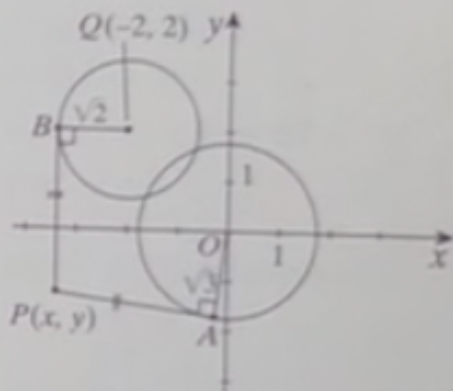
$$PA = PB$$

$$\sqrt{PO^2 - AO^2} = \sqrt{PQ^2 - BQ^2}$$

$$PO^2 - AO^2 = PQ^2 - BQ^2$$

$$(x^2 + y^2) - 3 = [(x+2)^2 + (y-2)^2] - 2$$

$$4x - 4y = -9$$



The equation of the locus of point $P(x, y)$ is $4x - 4y = -9$, excluding the part of the line that is inside either circle.

$$(2) \begin{cases} x^2 + y^2 = 9 \\ (x-3)^2 + (y+1)^2 = 4 \end{cases}$$

[Sol] Since the tangent lines from P are equal,

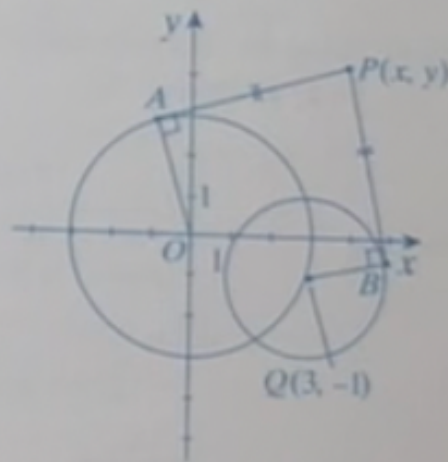
$$PA = PB$$

$$\sqrt{PO^2 - AO^2} = \sqrt{PQ^2 - BQ^2}$$

$$PO^2 - AO^2 = PQ^2 - BQ^2$$

$$(x^2 + y^2) - 9 = [(x-3)^2 + (y+1)^2] - 4$$

$$6x - 2y = 15$$

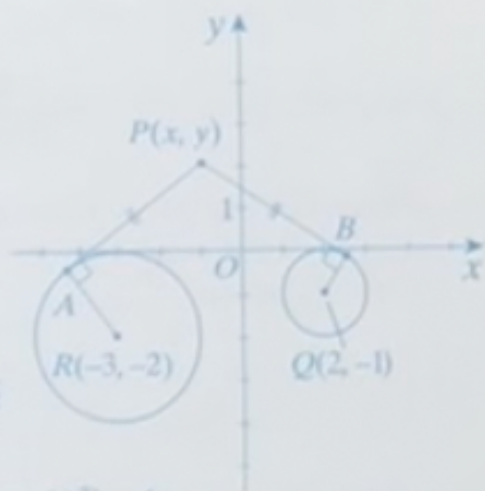


The equation of the locus of point $P(x, y)$ is $6x - 2y = 15$, excluding the part of the line that is inside either circle.

N 15 b

$$(3) \begin{cases} (x+3)^2 + (y+2)^2 = 4 \\ (x-2)^2 + (y+1)^2 = 1 \end{cases}$$

[Sol] Since the tangent lines from P are equal,



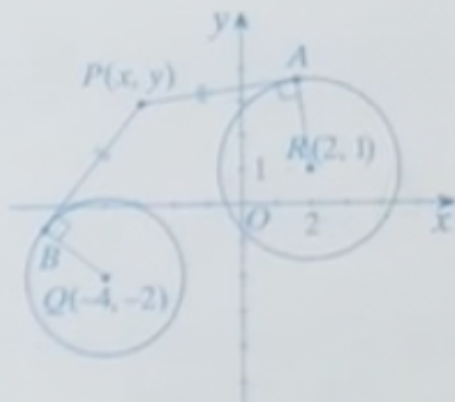
$$\begin{aligned} PA &= PB \\ \sqrt{PR^2 - AR^2} &= \sqrt{PQ^2 - BQ^2} \\ PR^2 - AR^2 &= PQ^2 - BQ^2 \end{aligned}$$

$$\begin{aligned} [(x+3)^2 + (y+2)^2] - 4 &= [(x-2)^2 + (y+1)^2] - 1 \\ 10x + 2y &= -5 \end{aligned}$$

The equation of the locus of point $P(x, y)$ is: $10x + 2y = -5$

$$(4) \begin{cases} (x-2)^2 + (y-1)^2 = 7 \\ (x+4)^2 + (y+2)^2 = 5 \end{cases}$$

[Sol] Since the tangent lines from P are equal,



$$\begin{aligned} PA &= PB \\ \sqrt{PR^2 - AR^2} &= \sqrt{PQ^2 - BQ^2} \\ PR^2 - AR^2 &= PQ^2 - BQ^2 \end{aligned}$$

$$\begin{aligned} [(x-2)^2 + (y-1)^2] - 7 &= [(x+4)^2 + (y+2)^2] - 5 \\ 12x + 6y &= -17 \end{aligned}$$

The equation of the locus of point $P(x, y)$ is: $12x + 6y = -17$

Time : to : Date Name

100%	90%	80%	
(mistakes) 0	—	—	—

1. Given that the distance between two moving points $A(a, 0)$ and $B(0, b)$ is p , obtain the equation of the locus of the midpoint Q of AB .

[Sol] Let (X, Y) be the coordinates of Q .

$$\begin{cases} X = \frac{a}{2} & \dots \textcircled{1} \\ Y = \frac{b}{2} & \dots \textcircled{2} \\ a^2 + b^2 = p^2 & \dots \textcircled{3} \end{cases}$$

Using $\textcircled{1}$ and $\textcircled{2}$,

$$a = 2X \quad \dots \textcircled{4}$$

$$b = 2Y \quad \dots \textcircled{5}$$

Substituting $\textcircled{4}$ and $\textcircled{5}$ into $\textcircled{3}$,

$$X^2 + Y^2 = \frac{p^2}{4}$$

$\therefore$ The equation of the locus of point Q is the circle with center $(0, 0)$ and radius $\frac{p}{2}$.

2. Given that the distance between two moving points $A(a, 0)$ and $B(0, b)$ is p , obtain the equation of the locus of center of gravity G of $\triangle OAB$.

[Sol] Let (X, Y) be the coordinates of G .

$$\begin{cases} X = \frac{a}{3} & \dots \textcircled{1} \\ Y = \frac{b}{3} & \dots \textcircled{2} \\ a^2 + b^2 = p^2 & \dots \textcircled{3} \end{cases}$$

Using $\textcircled{1}$ and $\textcircled{2}$,

$$a = 3X \quad \dots \textcircled{4}$$

$$b = 3Y \quad \dots \textcircled{5}$$

Substituting $\textcircled{4}$ and $\textcircled{5}$ into $\textcircled{3}$,

$$X^2 + Y^2 = \frac{p^2}{9}$$

$\therefore$ The equation of the locus of point G is the circle with center $(0, 0)$ and radius $\frac{p}{3}$.

N 16 b

3. Given that the circle $x^2 + y^2 - 6y - 16 = 0$ passes through points $A(-4, 0)$ and $B(4, 0)$, obtain the equation of the locus of the center of gravity G of $\triangle PAB$, where P is a point on the circle.

[Sol] Let (X, Y) and (x, y) be the coordinates of G and P , respectively.

$$\begin{cases} X = \frac{x}{3} & \dots \textcircled{1} \\ Y = \frac{y}{3} & \dots \textcircled{2} \\ x^2 + y^2 - 6y - 16 = 0 & \dots \textcircled{3} \end{cases}$$

Using $\textcircled{1}$ and $\textcircled{2}$,

$$x = 3X \quad \dots \textcircled{4}$$

$$y = 3Y \quad \dots \textcircled{5}$$

Substituting $\textcircled{4}$ and $\textcircled{5}$ into $\textcircled{3}$,

$$(3X)^2 + (3Y)^2 - 6(3Y) - 16 = 0$$

$$9X^2 + 9Y^2 - 18Y - 16 = 0$$

$$X^2 + (Y - 1)^2 = \frac{25}{9}$$

Therefore, the equation of the locus of point G is the circle with center $(0, 1)$ and radius $\frac{5}{3}$, excluding points $\left(\pm \frac{4}{3}, 0\right)$.

(Point P cannot coincide with A or B .)

Time : to : Date Name

100%	90%	80%	70%
(mistakes) 0	—	—	1

1. Given that line segment AB has length 5 with moving endpoint A on the x -axis and moving endpoint B on the y -axis, obtain the equation of the locus of point P which divides AB internally in the ratio of 3:2.

[Sol] Placing A , B , and P at $(a, 0)$, $(0, b)$, and (X, Y) ,

$$X = \frac{2}{5}a, \quad Y = \frac{3}{5}b$$

$$a = \frac{5}{2}X, \quad b = \frac{5}{3}Y \quad \dots \textcircled{1}$$

Since AB has length 5,

$$a^2 + b^2 = 25 \quad \dots \textcircled{2}$$

Substituting $\textcircled{1}$ into $\textcircled{2}$,

$$\left(\frac{5}{2}X\right)^2 + \left(\frac{5}{3}Y\right)^2 = 25$$

$$\frac{X^2}{4} + \frac{Y^2}{9} = 1$$

$\therefore$ The equation of the locus is the ellipse $\frac{X^2}{4} + \frac{Y^2}{9} = 1$.

2. Obtain the equation of the locus of the midpoint M of the line segment connecting the fixed point $A(4, -1)$ and a moving point P on the line $x + 2y - 3 = 0$.

[Sol] Let (X, Y) and (x, y) be the coordinates of M and P , respectively.

$$\begin{cases} X = \frac{x+4}{2} & \dots \textcircled{1} \end{cases}$$

$$\begin{cases} Y = \frac{y-1}{2} & \dots \textcircled{2} \end{cases}$$

$$\begin{cases} x + 2y - 3 = 0 & \dots \textcircled{3} \end{cases}$$

Using $\textcircled{1}$ and $\textcircled{2}$,

$$x = 2X - 4 \dots \textcircled{4} \quad \text{and} \quad y = 2Y + 1 \dots \textcircled{5}$$

Substituting $\textcircled{4}$ and $\textcircled{5}$ into $\textcircled{3}$,

$$2X + 4Y - 5 = 0$$

$\therefore$ The equation of the locus is the line $2X + 4Y - 5 = 0$.

N 17 b

3. Dropping a perpendicular from point Q on circle $x^2 + y^2 = 4$ to the x -axis at point H , and placing point P on QH , such that P divides QH internally in the ratio of 1 : 1, obtain the equation of the locus of point P .

[Sol] Let (x, y) be the coordinates of point Q on $x^2 + y^2 = 4$... ①
and let (X, Y) be the coordinates of P .

$$X = x, \quad Y = \frac{1}{2}y$$

$$\therefore x = X, \quad y = 2Y$$

Substituting these into ①,

$$\frac{X^2}{4} + Y^2 = 1$$

$\therefore$ The equation of the locus is the ellipse $\frac{X^2}{4} + Y^2 = 1$.

4. Dropping a perpendicular from point Q on circle $x^2 + y^2 = 16$ to the x -axis at point H , and placing point P on QH such that $PH = \frac{1}{4}QH$, obtain the equation of the locus of point P .

[Sol] Let (x, y) be the coordinates of point Q on $x^2 + y^2 = 16$... ①
and let (X, Y) be the coordinates of P .

$$X = x, \quad Y = \frac{1}{4}y$$

$$\therefore x = X, \quad y = 4Y$$

Substituting these into ①,

$$\frac{X^2}{16} + Y^2 = 1$$

$\therefore$ The equation of the locus is the ellipse $\frac{X^2}{16} + Y^2 = 1$.

Loci 2

Time : to : Date Name

100%	90%	80%
(mistakes) 0	-	-

1. Letting P and Q be the two points of intersection of a line parallel to line $y = x$ with the parabola $y^2 = 2x$, obtain the equation of the locus of the midpoint of line segment PQ .

[Sol] $\begin{cases} y = x + a & \dots \textcircled{1} \\ y^2 = 2x & \dots \textcircled{2} \end{cases}$

Substituting $\textcircled{1}$ into $\textcircled{2}$,

$$(x + a)^2 = 2x$$

$$\therefore x^2 + \boxed{2(a-1)}x + a^2 = 0 \dots \textcircled{3}$$

Letting (x_1, y_1) and (x_2, y_2) be the coordinates of P and Q ,
 x_1 and x_2 are the real solutions of $\textcircled{3}$.

In order for $\textcircled{3}$ to have two different solutions,

$$D > 0$$

$$-2a + 1 > 0$$

$$a < \boxed{\frac{1}{2}} \dots \textcircled{4}$$

Letting $M(X, Y)$ be the midpoint of PQ ,

$$X = \frac{x_1 + x_2}{2}$$

Applying the Root-Coefficient Relationship to $\textcircled{3}$,

$$x_1 + x_2 = \boxed{-2(a-1)}$$

$$\therefore X = \boxed{-(a-1)} \dots \textcircled{5}$$

Since point M lies on line $\textcircled{1}$,

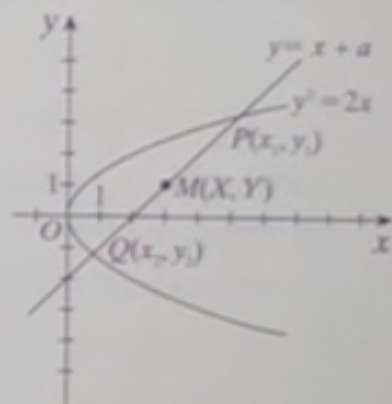
$$Y = X + a = \boxed{1}$$

Obtaining the range of X from $\textcircled{4}$ and $\textcircled{5}$,

$$X > \boxed{\frac{1}{2}}$$

Therefore, the equation of the locus is:

$$Y = \boxed{1}, \left(X > \boxed{\frac{1}{2}} \right)$$



N 18 b

2. Letting P and Q be the two points of intersection of the parabola $y = x^2$ and a line with slope 2, obtain the equation of the locus of the midpoint M of line segment PQ .

$$[\text{Sol}] \begin{cases} y = 2x + a & \dots \textcircled{1} \\ y = x^2 & \dots \textcircled{2} \end{cases}$$

Substituting $\textcircled{1}$ into $\textcircled{2}$,

$$2x + a = x^2$$

$$\therefore x^2 - 2x - a = 0 \quad \dots \textcircled{3}$$

Letting (x_1, y_1) and (x_2, y_2) be the coordinates of P and Q ,
 x_1 and x_2 are the real solutions of $\textcircled{3}$.

In order for $\textcircled{3}$ to have two different solutions,

$$D > 0$$

$$1 + a > 0$$

$$\therefore a > -1 \quad \dots \textcircled{4}$$

Letting $M(X, Y)$ be the midpoint of PQ ,

$$X = \frac{x_1 + x_2}{2}$$

Applying the Root-Coefficient Relationship to $\textcircled{3}$,

$$x_1 + x_2 = 2$$

$$\therefore X = 1$$

Since point M lies on line $\textcircled{1}$,

$$Y = 2X + a = 2 + a \quad \dots \textcircled{5}$$

From $\textcircled{4}$ and $\textcircled{5}$, $Y > 1$.

Therefore, the equation of the locus is: $X = 1, (Y > 1)$

Time : to : Date Name

100%	90%	80%	70%
(mistakes) 0	-	-	1

1. Obtain the equation of the locus of the points of intersection of each the following pairs of lines for all real values of m .

$$(1) \begin{cases} 4y + mx = -3m & \dots \textcircled{1} \\ my - 2x = -2 & \dots \textcircled{2} \end{cases}$$

[Sol] Rewriting ①,

$$m(x + 3) = -4y$$

- (a) When $x \neq -3$,

$$m = -\frac{4y}{x+3}$$

Substituting this into ②,

$$\frac{(x+1)^2}{4} + \frac{y^2}{2} = 1$$

- (b) When $x = -3$,

$$\textcircled{1} \text{ gives } y = 0,$$

But $(x, y) = (-3, 0)$ does not satisfy ②.

$\therefore$ The equation of the locus obtained from (a) and (b) is the ellipse

$$\frac{(x+1)^2}{4} + \frac{y^2}{2} = 1, \text{ excluding the point } (-3, 0).$$

$$(2) \begin{cases} mx + y = 2m & \dots \textcircled{1} \\ 2x - my = -8 & \dots \textcircled{2} \end{cases}$$

[Sol] Rewriting ①,

$$m(x - 2) = -y$$

- (a) When $x \neq 2$,

$$m = -\frac{y}{x-2}$$

Substituting this into ②,

$$\frac{(x+1)^2}{9} + \frac{y^2}{18} = 1$$

- (b) When $x = 2$,

$$\textcircled{1} \text{ gives } y = 0,$$

But $(x, y) = (2, 0)$ does not satisfy ②.

$\therefore$ The equation of the locus obtained from (a) and (b) is the ellipse

$$\frac{(x+1)^2}{9} + \frac{y^2}{18} = 1, \text{ excluding the point } (2, 0).$$

N 19 b

2. Given the two lines: $mx + y = 2 \dots \textcircled{1}$ and $x - m(y - 2) = 3 \dots \textcircled{2}$:

(1) Obtain the equation of the locus of their points of intersection for all real values of m .

[Sol] $\textcircled{1}$ gives $mx = -y + 2 \dots \textcircled{3}$

(a) When $x \neq 0$, $\textcircled{3}$ gives $m = \frac{-y+2}{x}$.

Substituting this into $\textcircled{2}$ and rearranging,

$$\left(x - \frac{3}{2}\right)^2 + (y - 2)^2 = \frac{9}{4}, \quad x \neq 0$$

(b) When $x = 0$, $\textcircled{3}$ gives $y = 2$.

But these values do not satisfy $\textcircled{2}$.

Thus, point $(0, 2)$ is not in the locus.

Therefore, the equation of the locus is the circle with center $\left(\frac{3}{2}, 2\right)$ and radius $\frac{3}{2}$, excluding the point $(0, 2)$.

(2) Draw the path traced out by the points of intersection of the two lines when $0 \leq m \leq 1$.

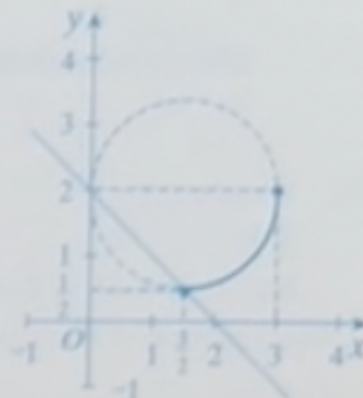
[Sol] $\textcircled{1}$ is a line with slope $-m$ passing through point $(0, 2)$.

When $m = 1$, solving $y = -x + 2$ and $\left(x - \frac{3}{2}\right)^2 + (y - 2)^2 = \frac{9}{4}$.

The point of intersection with the circle obtained in (1) is: $\left(\frac{3}{2}, \frac{1}{2}\right)$

When $m = 0$, the line becomes $y = 2$, and its point of intersection with the circle is $(3, 2)$.

Thus, when $0 \leq m \leq 1$, the locus is the solid part of the circle in the diagram.



Time : to : Date Name

100%	90%	80%
(mistakes) 0	-	-

1. Obtain the equation of the locus of the midpoint, M , of the line segment connecting the fixed point $A(-2, -5)$ and a moving point P on the line $2x + 4y - 1 = 0$.

[Sol] Let (X, Y) and (x, y) be the coordinates of M and P , respectively.

$$\begin{cases} X = \frac{x-2}{2} & \dots \textcircled{1} \\ Y = \frac{y-5}{2} & \dots \textcircled{2} \\ 2x + 4y - 1 = 0 & \dots \textcircled{3} \end{cases}$$

Using $\textcircled{1}$ and $\textcircled{2}$,

$$x = 2X + 2 \quad \dots \textcircled{4}$$

$$y = 2Y + 5 \quad \dots \textcircled{5}$$

Substituting $\textcircled{4}$ and $\textcircled{5}$ into $\textcircled{3}$,

$$4X + 8Y + 23 = 0$$

$\therefore$ The equation of the locus is: $4X + 8Y + 23 = 0$

2. Obtain the equation of the locus of point $P(x, y)$ such that the lengths of the tangent lines from P to the following pair of circles is equal.

$$\begin{cases} (x+1)^2 + (y+2)^2 = 3 \\ (x-4)^2 + \left(y - \frac{1}{2}\right)^2 = 2 \end{cases}$$

[Sol] Since the tangent lines from P are equal,

$$PA = PB$$

$$\sqrt{PR^2 - AR^2} = \sqrt{PQ^2 - BQ^2}$$

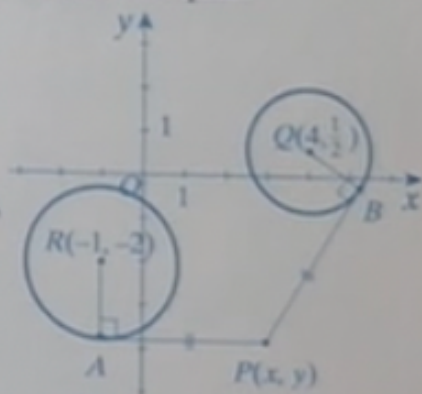
$$PR^2 - AR^2 = PQ^2 - BQ^2$$

$$[(x+1)^2 + (y+2)^2] - 3 = [(x-4)^2 + \left(y - \frac{1}{2}\right)^2] - 2$$

$$10x + 5y = \frac{49}{4}$$

The equation of the locus is: $10x + 5y = \frac{49}{4}$

$$[40x + 20y = 49]$$



N 20 b

3. Given that the circle $x^2 - 4x + y^2 - 2y = 3$ passes through points $A(0, -1)$ and $B(4, -1)$, obtain the equation of the locus of the center of gravity G of $\triangle PAB$, where P is a point on the circle.

[Sol] Let (X, Y) and (x, y) be the coordinates of G and P , respectively.

$$\begin{cases} X = \frac{4+x}{3} & \dots \textcircled{1} \\ Y = \frac{-2+y}{3} & \dots \textcircled{2} \\ x^2 - 4x + y^2 - 2y = 3 & \dots \textcircled{3} \end{cases}$$

Using $\textcircled{1}$ and $\textcircled{2}$,

$$x = 3X - 4 \quad \dots \textcircled{4}$$

$$y = 3Y + 2 \quad \dots \textcircled{5}$$

Substituting $\textcircled{4}$ and $\textcircled{5}$ into $\textcircled{3}$,

$$(3X - 4)^2 - 4(3X - 4) + (3Y + 2)^2 - 2(3Y + 2) = 3$$

$$9X^2 - 36X + 9Y^2 + 6Y = -29$$

$$X^2 - 4X + Y^2 + \frac{2}{3}Y = -\frac{29}{9}$$

$$(X - 2)^2 + \left(Y + \frac{1}{3}\right)^2 = \frac{8}{9}$$

Therefore, the equation of the locus of point G is the circle

with center $\left(2, -\frac{1}{3}\right)$ and radius $\frac{2\sqrt{2}}{3}$, excluding points

$\left(\frac{4}{3}, -1\right)$ and $\left(\frac{8}{3}, -1\right)$.

(Point P cannot coincide with A or B .)

Quadratic Inequalities and Regions

Time : to : Date Name

100%	90%	80%	70%
(mistakes) 0	-	1	-

In each of the following exercises, graph the region representing the inequality.

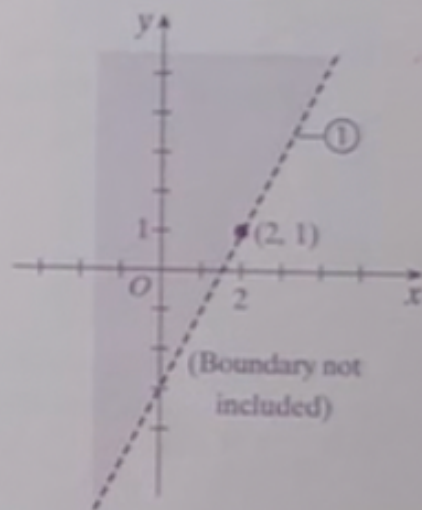
Ex. $y > 2x - 3$

[Sol] Begin by drawing the graph of the equality $y = 2x - 3$... ①

Line ① is the boundary of the graphed region.

Consider the values of x at which:
 $y > 2x - 3$
 (i.e. greater than the y -values of ①.)

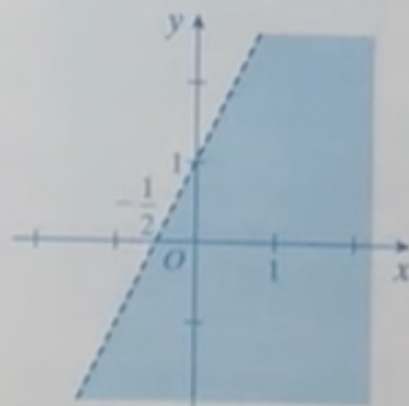
The shaded region represents all the (x, y) pairs that satisfy this condition of the given inequality.



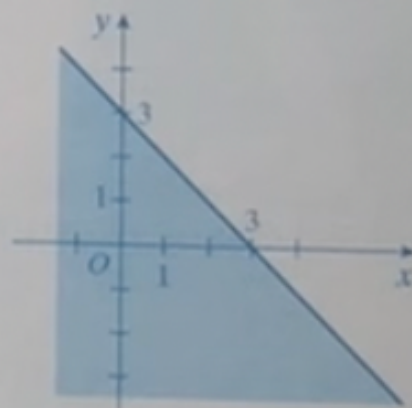
*Note that the boundary, line ①, is dashed because the coordinate pairs on this line are not included in the solution. When the boundary is included in the solution, it is graphed as a solid line.

(1) $y < 2x + 1$ (dashed line)

(2) $y \leq -x + 3$ (solid line)



The shaded region above
(not including the boundary)



The shaded region above
(including the boundary)

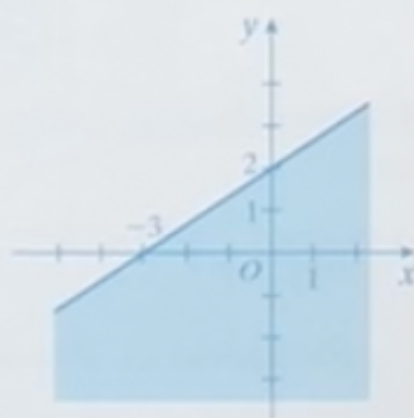
N 21 b

(3) $2x + y - 4 > 0$



The shaded region above
(not including the boundary)

(5) $2x - 3y + 6 \geq 0$



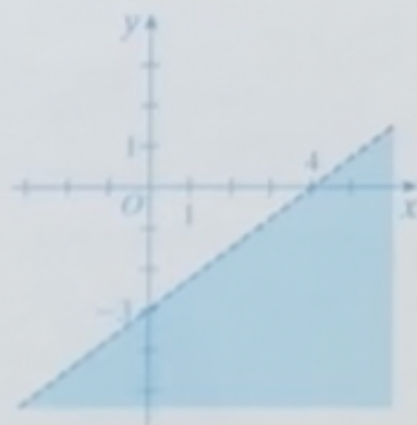
The shaded region above
(including the boundary)

(4) $2x - y - 4 > 0$



The shaded region above
(not including the boundary)

(6) $3x - 4y > 12$



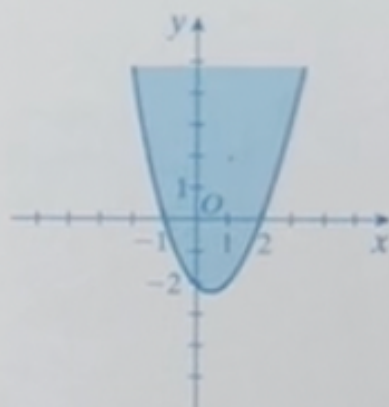
The shaded region above
(not including the boundary)

Time : to : Date Name

100%	90%	80%	70%	60%
(mistake) 0	-	1	2	3

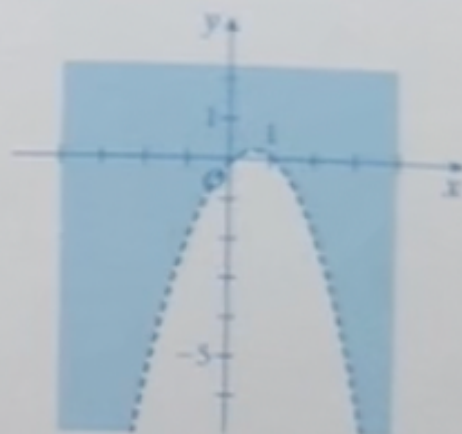
In each of the following exercises, graph the region representing the given inequality.

(1) $y \geq x^2 - x - 2$



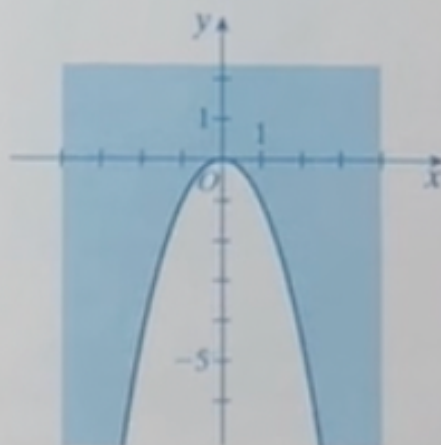
The shaded region above
(including the boundary)

(3) $x^2 > x - y$



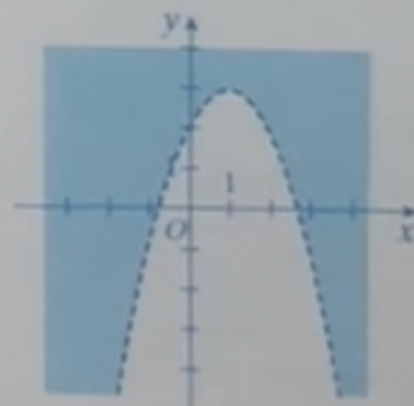
The shaded region above
(not including the boundary)

(2) $x^2 + y \geq 0$



The shaded region above
(including the boundary)

(4) $y + (x - 1)^2 > 3$

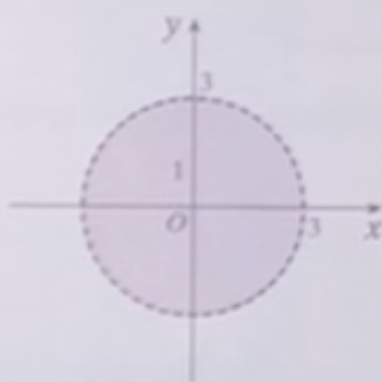


The shaded region above
(not including the boundary)

N 22 b

Ex.

$$x^2 + y^2 < 9$$



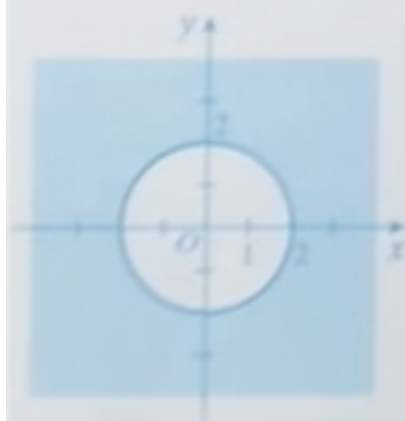
The shaded region above
(not including the boundary)

$$(6) \quad x^2 + y^2 < 3x$$



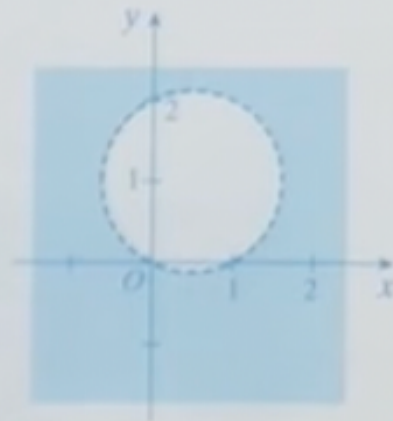
The shaded region above
(not including the boundary)

$$(5) \quad x^2 + y^2 \geq 4$$



The shaded region above
(including the boundary)

$$(7) \quad x^2 + y^2 > x + 2y$$



The shaded region above
(not including the boundary)

Quadratic Inequalities and Regions

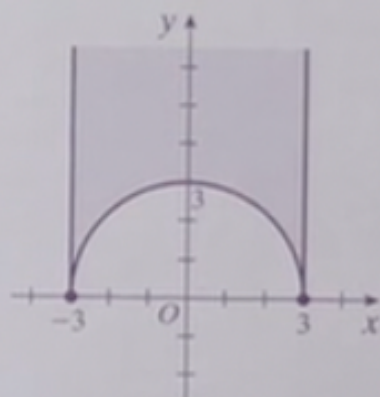
Time : to : Date Name

100%	90%	80%	70%
(mistakes) 0	-	1	-

In each of the following exercises, graph the region representing the given inequality.

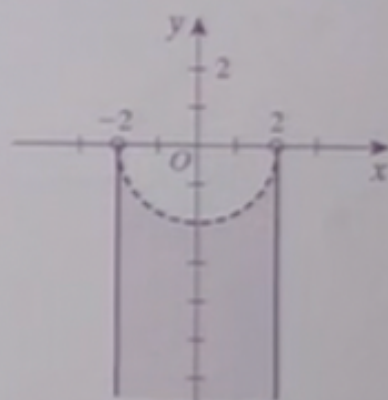
Ex. 1

$$y \geq \sqrt{9 - x^2}$$



Ex. 2

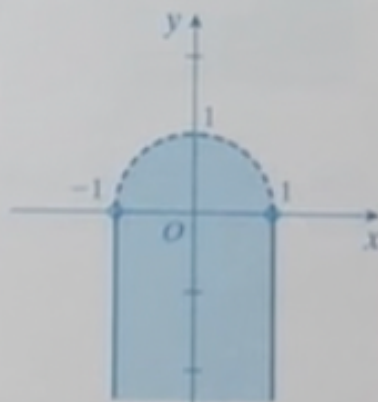
$$y < -\sqrt{4 - x^2}$$

**Note**

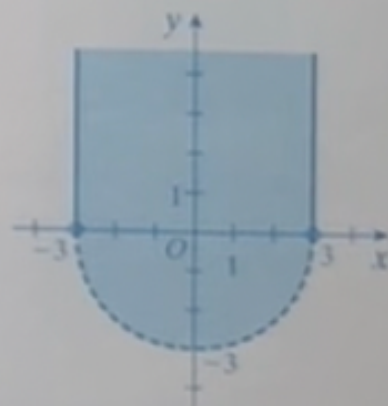
A closed circle, $\bullet$, is used to represent a point that is included in the solution.
An open circle, $\circ$, is used to represent a point that is not included in the solution.

$$(1) \quad y < \sqrt{1 - x^2}$$

$$(2) \quad y > -\sqrt{9 - x^2}$$



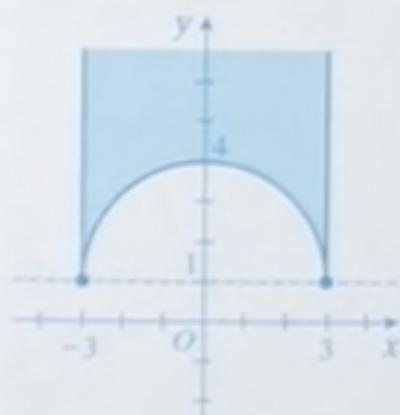
The shaded region above
(not including the dashed curve)



The shaded region above
(not including the dashed curve)

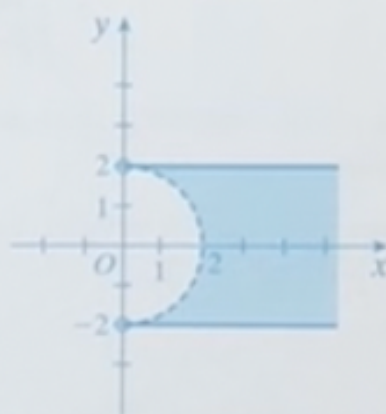
N 23 b

(3) $y \geq 1 + \sqrt{9 - x^2}$



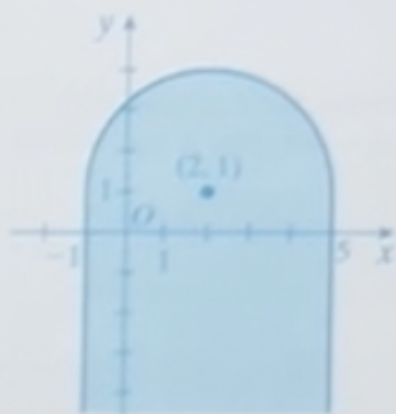
The shaded region above
(including the boundary)

(5) $x > \sqrt{4 - y^2}$



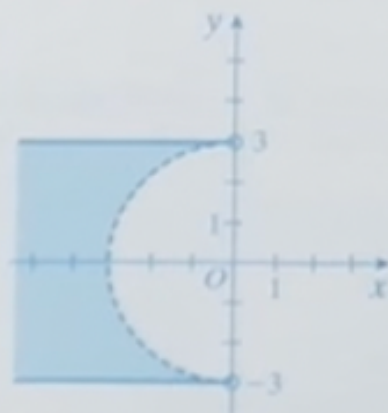
The shaded region above
(not including the dashed curve)

(4) $y \leq 1 + \sqrt{5 + 4x - x^2}$



The shaded region above
(including the boundary)

(6) $x < -\sqrt{9 - y^2}$



The shaded region above
(not including the dashed curve)

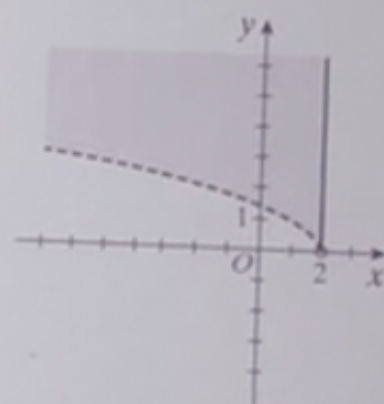
Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	1	-	2

In each of the following exercises, graph the region representing the given inequality.

Ex. 1

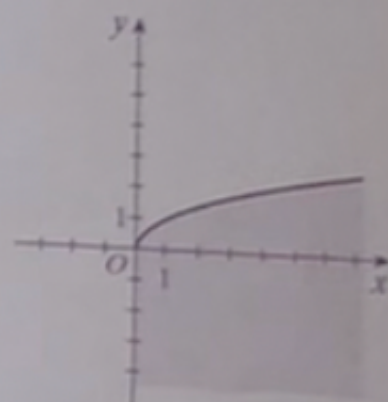
$$y > \sqrt{2-x}$$



The shaded region above
(not including the dashed curve)

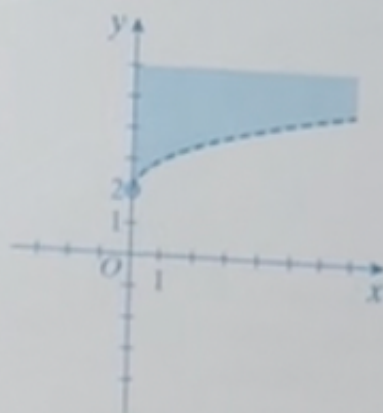
Ex. 2

$$y \leq \sqrt{x}$$



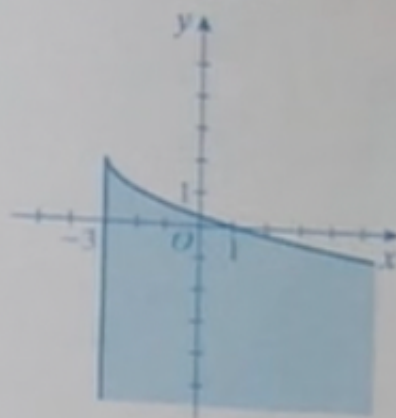
The shaded region above
(including the boundary)

(1) $y > 2 + \sqrt{x}$



The shaded region above
(not including the dashed curve)

(2) $y \leq 2 - \sqrt{3+x}$



The shaded region above
(including the boundary)

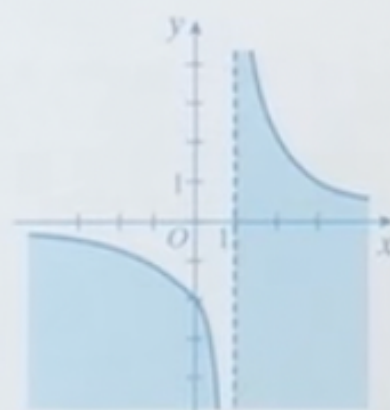
N 24 b

$$(3) \quad y > \frac{1}{x}$$



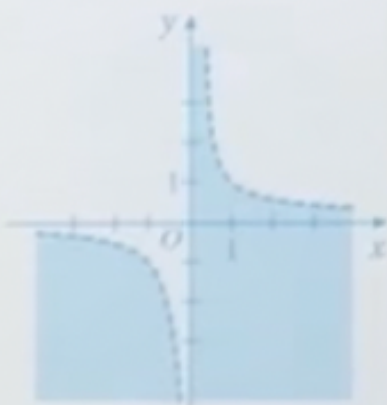
The shaded regions above
(not including the y-axis
and dashed curves)

$$(5) \quad y \leq \frac{2}{x-1}$$



The shaded regions above
(not including the dashed line)

$$(4) \quad y < \frac{1}{x}$$



The shaded regions above
(not including the y-axis and
dashed curves)

$$(6) \quad y > \frac{x+1}{x-1}$$



The shaded regions above
(not including the dashed line
and curves)

N 25 a

Quadratic Inequalities and Regions

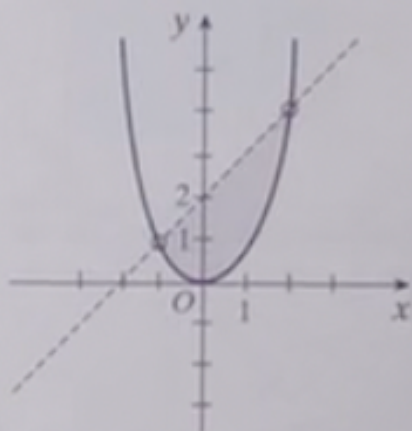
Time : to : Date Name

100%	90%	80%	70%	60%~
(mistakes) 0	-	1	-	2-

In each of the following exercises, graph the region satisfying both of the given inequalities.

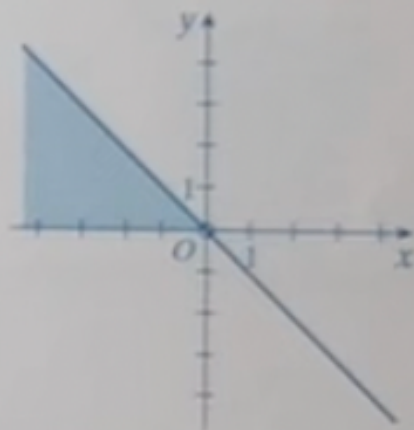
Ex.

$$\begin{cases} y \geq x^2 \\ y < x + 2 \end{cases}$$



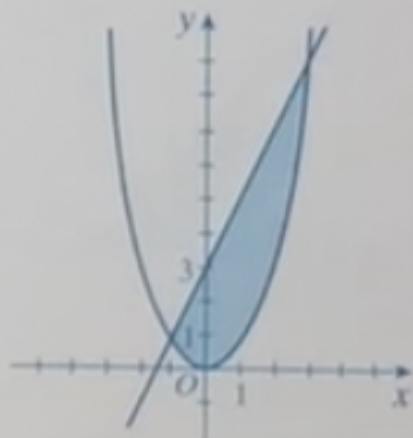
The shaded region above
(not including the dashed
line)

$$(2) \begin{cases} x + y \leq 0 \\ y > 0 \end{cases}$$



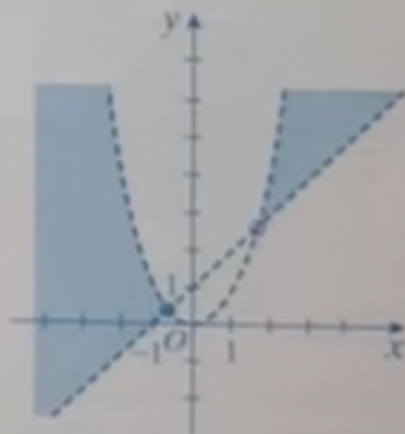
The shaded region above
(not including the x-axis)

$$(1) \begin{cases} 2x - y + 3 \geq 0 \\ y \geq x^2 \end{cases}$$



The shaded region above
(including the boundaries)

$$(3) \begin{cases} y > x + 1 \\ y < x^2 \end{cases}$$



The shaded regions above
(not including the boundaries)

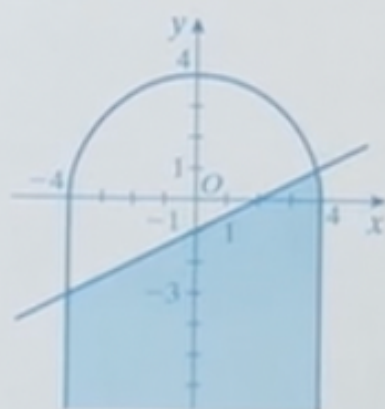
25 b

$$\begin{cases} x + y \geq 0 \\ x^2 + y^2 \leq 1 \end{cases}$$



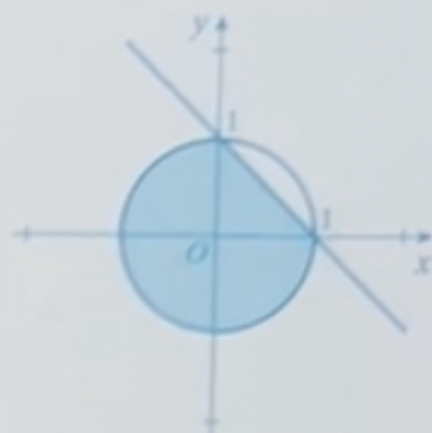
The shaded region above
(including the boundaries)

$$(6) \begin{cases} x - 2y \geq 2 \\ y \leq \sqrt{16 - x^2} \end{cases}$$



The shaded region above
(including the boundaries)

$$\begin{cases} x + y \leq 1 \\ x^2 + y^2 \leq 1 \end{cases}$$



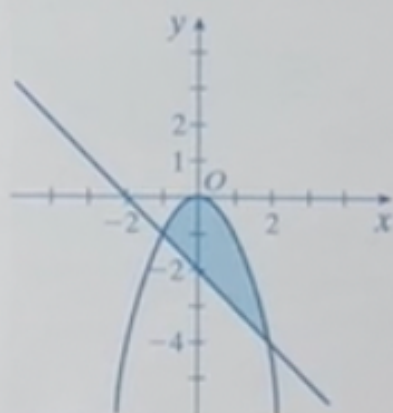
The shaded region above
(including the boundaries)

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	1	2	3	4

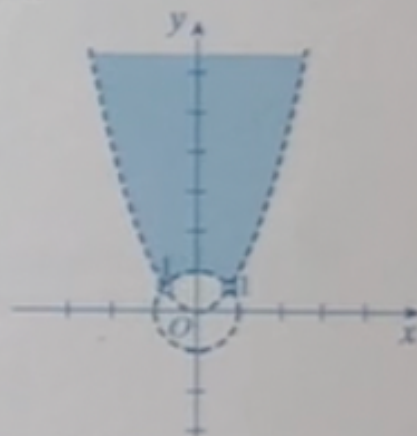
In each of the following exercises, graph the region satisfying the given system of inequalities.

(1)
$$\begin{cases} x^2 + y \leq 0 \\ x + y + 2 \geq 0 \end{cases}$$



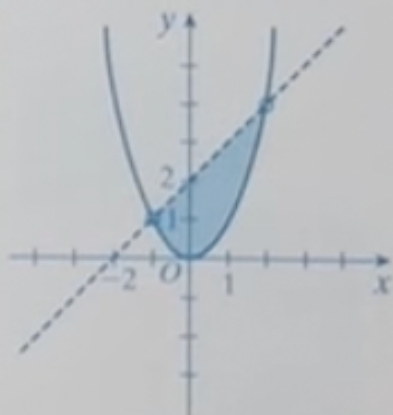
The shaded region above
(including the boundaries)

(3)
$$\begin{cases} x^2 + y^2 > 1 \\ y > x^2 \end{cases}$$



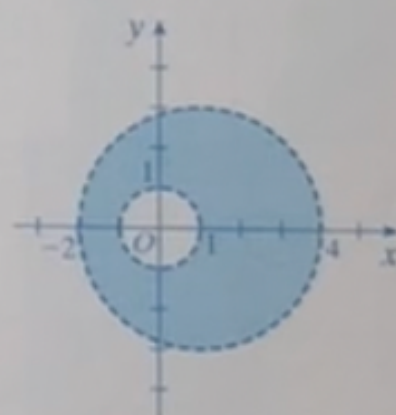
The shaded region above
(not including the boundaries)

(2)
$$x + 2 > y \geq x^2$$



The shaded region above
(not including the dashed line)

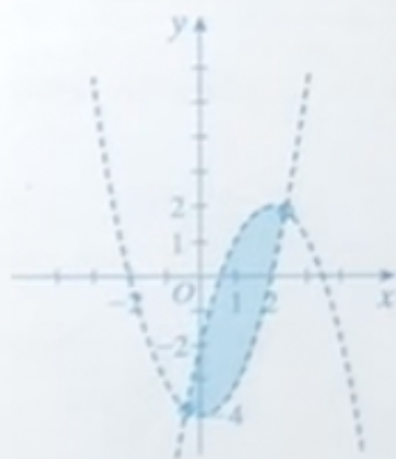
(4)
$$\begin{cases} x^2 + y^2 > 1 \\ (x - 1)^2 + y^2 < 9 \end{cases}$$



The shaded region above
(not including the boundaries)

N 26 b

$$(5) \begin{cases} y > x^2 - 4 \\ y + (x - 2)^2 < 2 \end{cases}$$



The shaded region above
(not including the boundaries)

$$(7) \begin{cases} |y| \geq 1 \\ x^2 + y^2 < 4 \end{cases}$$



The shaded regions above
(not including the dashed
curves)

$$(6) \begin{cases} x^2 > x + y \\ x + y > 0 \end{cases}$$



The shaded regions above
(not including the boundaries)

Time : to : Date Name

100%	90%	80%	70%	60%~
(mistakes) 0	-	1	-	2-

In each of the following exercises, graph the region representing the given inequality.

Ex.

$$(x - 1)(y - 3) < 0$$

[Sol] Begin by drawing the graphs of the equations,

$$x - 1 = 0 \quad \text{and} \quad y - 3 = 0$$

which represent the equations of two lines,

$$x = 1 \quad \text{and} \quad y = 3$$

The original inequality

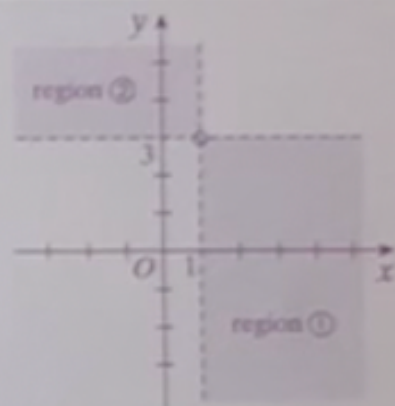
$$(x - 1)(y - 3) < 0 \text{ will hold true:}$$

$$\text{when } x > 1 \text{ and } y < 3 \dots \textcircled{1}$$

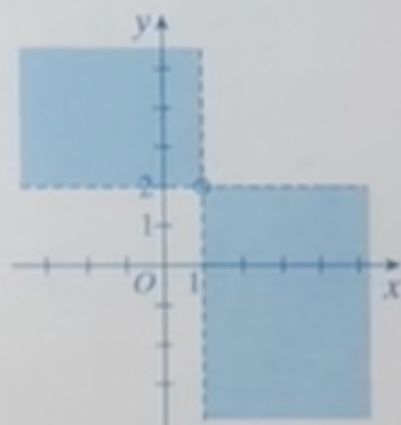
$$\text{OR when } x < 1 \text{ and } y > 3 \dots \textcircled{2}$$

The shaded regions ① and ② represent all the (x, y) pairs that satisfy these two conditions.

*Note that the graphs of the two lines are dashed because the points on these lines are not included in the solution.

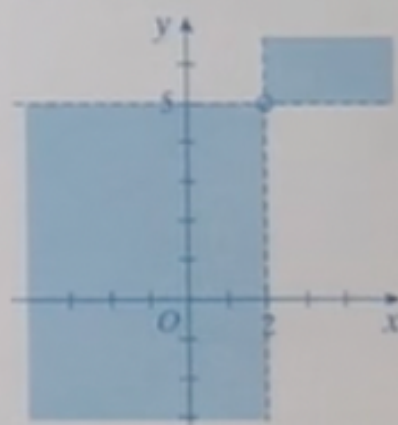


$$(1) \quad (x - 1)(y - 2) < 0$$



The shaded regions above
(not including the boundaries)

$$(2) \quad (x - 2)(y - 5) > 0$$



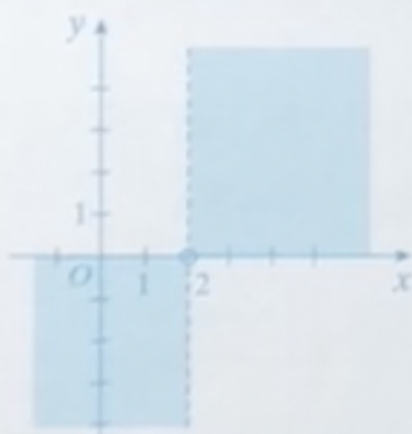
The shaded regions above
(not including the boundaries)

N 27 b

(3) $y(x - 2) > 0$

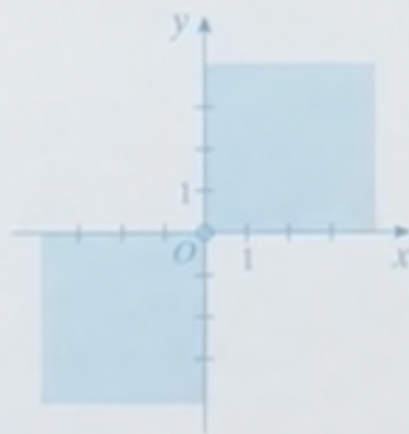
[Sol] The two lines are:

$y = 0$ and $x = 2$



The shaded regions above
(not including the x -axis
and dashed line)

(5) $xy > 0$



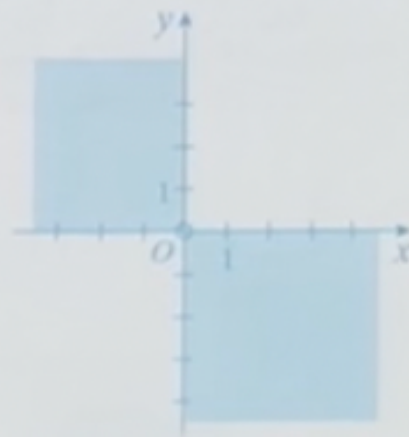
The shaded regions above
(not including the x and
 y -axes)

(4) $y(x - y) > 0$



The shaded regions above
(not including the x -axis
and dashed line)

(6) $xy < 0$



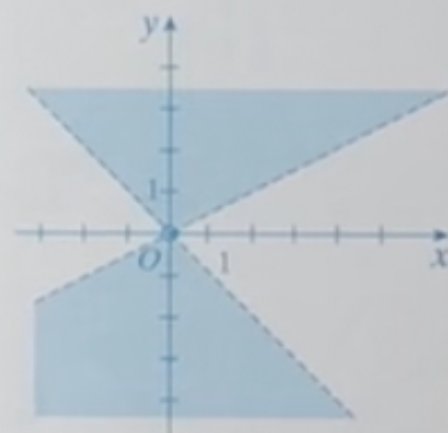
The shaded regions above
(not including the x and
 y -axes)

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	—	1	2	3~

In each of the following exercises, graph the region representing the given inequality.

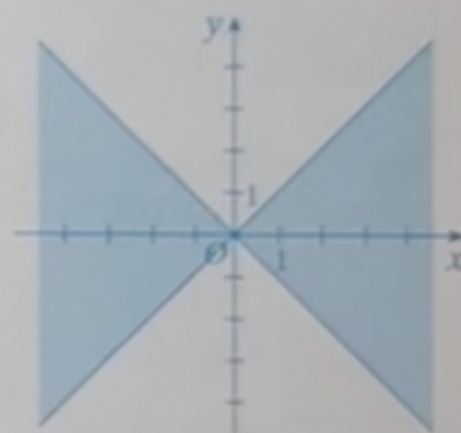
(1) $(x + y)(x - 2y) < 0$



The shaded regions above
(not including the boundaries)

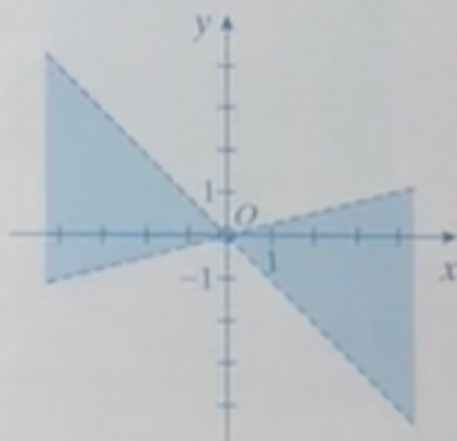
(3) $x^2 - y^2 \geq 0$

[Sol] $(x + y)(x - y) \geq 0$



The shaded regions above
(including the boundaries)

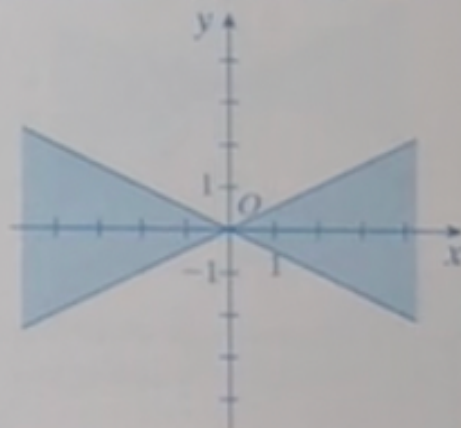
(2) $(x + y)(x - 4y) > 0$



The shaded regions above
(not including the boundaries)

(4) $x^2 - 4y^2 \geq 0$

[Sol] $(x + 2y)(x - 2y) \geq 0$

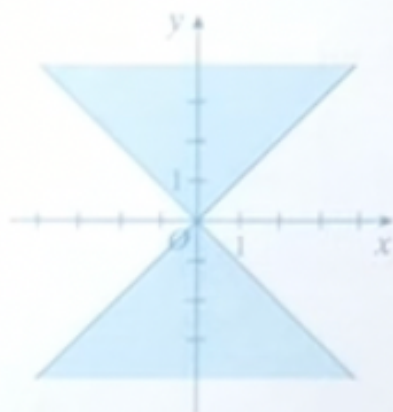


The shaded regions above
(including the boundaries)

N 28 b

(5) $y^2 - x^2 \geq 0$

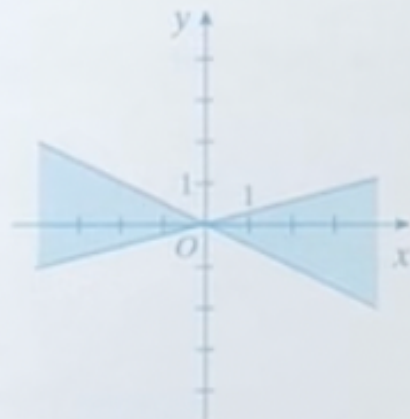
[Sol] $(y + x)(y - x) \geq 0$



The shaded regions above
(including the boundaries)

(7) $x^2 - 2xy - 8y^2 \geq 0$

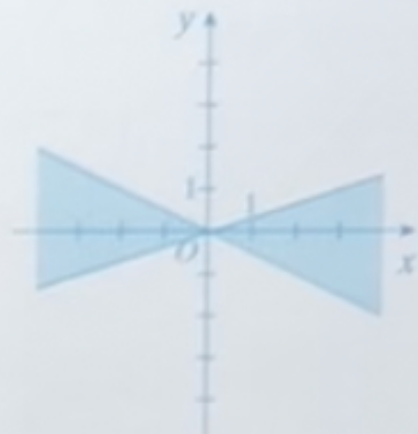
[Sol] $(x + 2y)(x - 4y) \geq 0$



The shaded regions above
(including the boundaries)

(6) $x^2 - xy - 6y^2 \geq 0$

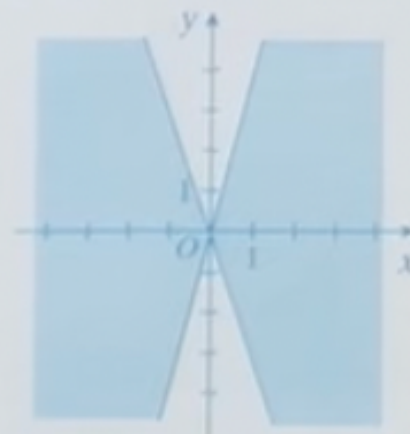
[Sol] $(x + 2y)(x - 3y) \geq 0$



The shaded regions above
(including the boundaries)

(8) $y^2 - xy - 12x^2 \leq 0$

[Sol] $(y - 4x)(y + 3x) \leq 0$



The shaded regions above
(including the boundaries)

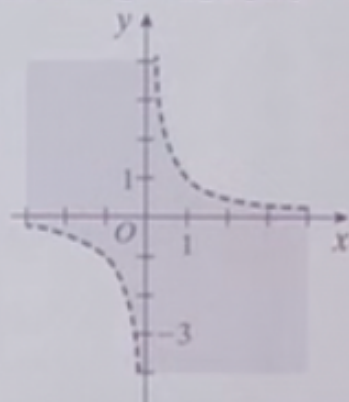
Time : to : Date Name

100%	90%	80%	70%	60%~
(mistakes) 0	—	1	2	3—

In each of the following exercises, graph the region representing the given inequality.

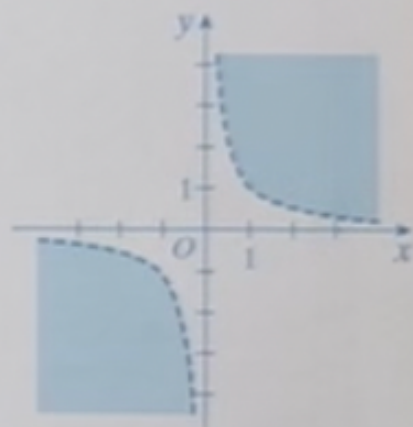
Ex.

$xy < 1$

[Sol] Drawing the graph $y = \frac{1}{x}$.

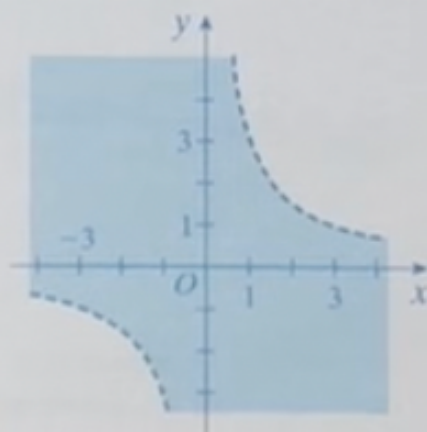
The shaded region above
(not including the boundaries)

(2) $xy - 1 > 0$



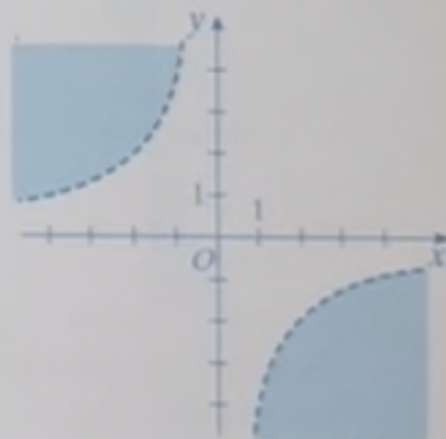
The shaded regions above
(not including the boundaries)

(1) $xy < 3$



The shaded region above
(not including the boundaries)

(3) $xy + 4 < 0$

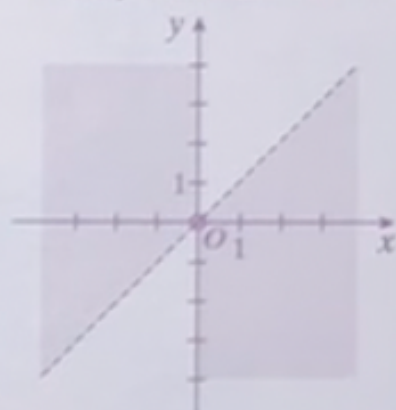


The shaded regions above
(not including the boundaries)

N 29 b

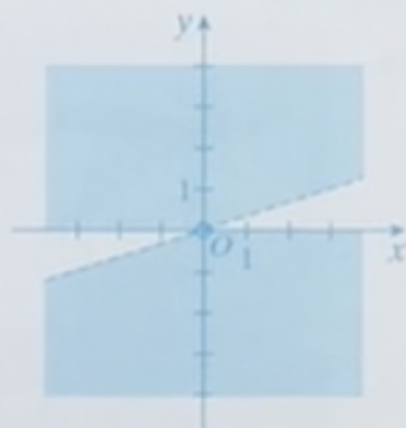
Ex. $\frac{y}{x} < 1$

[Sol] Graphing this fractional function, note that the asymptotes are: the y-axis and line $x = y$



The shaded regions above (not including the y-axis and dashed line)

(5) $\frac{x}{y} < 3$



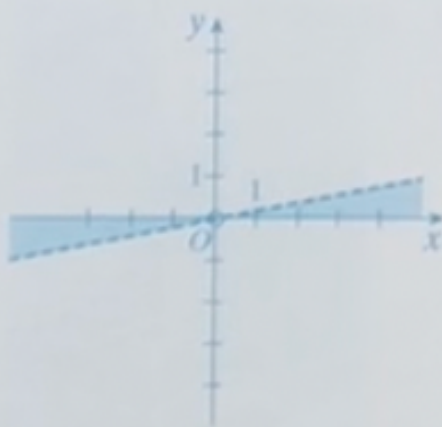
The shaded regions above (not including the x-axis and dashed line)

(4) $\frac{y}{x} > 3$



The shaded regions above (not including the y-axis and dashed line)

(6) $\frac{x}{y} > 5$



The shaded regions above (not including the x-axis and dashed line)

Quadratic Inequalities and Regions

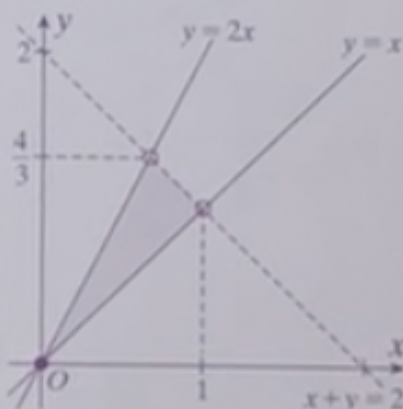
Time : to : Date Name

100%	90%	80%	70%	60%~
(mistakes) 0	-	1	-	2-

In each of the following exercises, graph the region satisfying the given system of inequalities, and then state the range of values of x and y .

Ex.

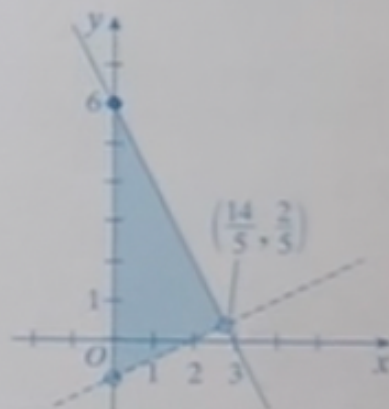
$$\begin{cases} y \geq x \\ y \leq 2x \\ x + y < 2 \end{cases}$$



From the coordinates of the points of intersection,

$$0 \leq x < 1, \quad 0 \leq y < \frac{4}{3}$$

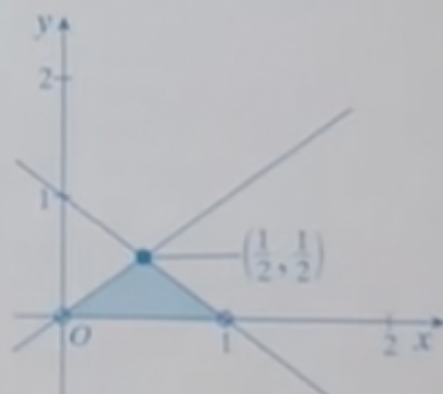
$$(2) \begin{cases} x \geq 0 \\ 2x + y \leq 6 \\ x - 2y < 2 \end{cases}$$



The shaded region above (not including the dashed line)

$$0 \leq x < \frac{14}{5}, \quad -1 < y \leq 6$$

$$(1) \begin{cases} y > 0 \\ y \leq x \\ x + y \leq 1 \end{cases}$$

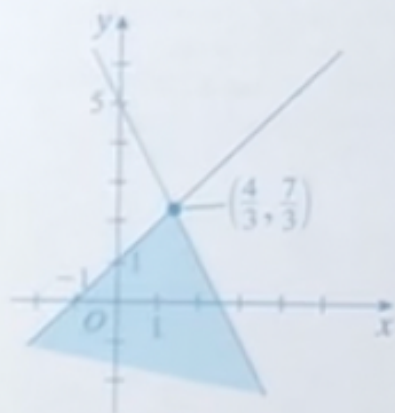


The shaded region above (not including the x -axis)

$$0 < x < 1, \quad 0 < y \leq \frac{1}{2}$$

N 30 b

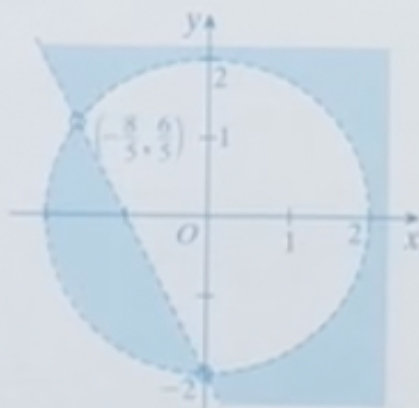
$$(3) \begin{cases} x - y + 1 \geq 0 \\ 5 - 2x - y \geq 0 \end{cases}$$



The shaded region above
(including the boundaries)

$$\text{All real } x, y \leq \frac{7}{3}$$

$$(5) (x^2 + y^2 - 4)(2x + y + 2) > 0$$



The shaded regions above
(not including the boundaries)

$$\text{All real } x \text{ and } y$$

$$(4) \begin{cases} x^2 + y^2 > 4 \\ 2x + y + 2 > 0 \end{cases}$$



The shaded region above
(not including the boundaries)

$$\text{All real } x \text{ and } y$$

Arithmetic Sequences

Time : to : Date Name

100%	90%	80%	70%	60%~
(mistakes) 0	1-2	3-4	5-6	7-

A succession of numbers which are arranged in a specific order according to a rule is called a **sequence**.

1. Fill in each of the following boxes with the appropriate number for the given sequence.

(1) 1, 3, 5, , 9, 11, 13, ...

(6) 3, 9, 27, 81, , 729, ...

(2) 2, 4, 6, 8, , 12, 14, ...

(7) $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \frac{9}{10}, \frac{11}{12}, \frac{13}{14}, \dots$

(3) 1, 4, 7, 10, , 16, 19, ...

(8) 4, 2, , -2, , -6, -8, ...

(4) 2, 5, 8, , 14, 17, ...

(9) 1, 4, 9, 16, , 36, 49, ...

(5) 2, 4, 8, , 32, 64, 128, ...

(10) 2, 6, 12, 20, , 42, 56, ...

Each number in a sequence is called a **term** of that sequence, and in order from the beginning, they are called the *first term*, *second term*, *third term*, and so on.

2. Fill in each box with the corresponding term from the sequences of the above exercises (1)-(10).

(1) In exercise (2), the 6th term is .

(2) In exercise (5), the 2nd term is .

(3) In exercise (6), the 1st term is .

(4) In exercise (7), the 4th term is .

(5) In exercise (8), the 7th term is .

(6) In exercise (10), the 3rd term is .

N 31 b

The position of each term in a sequence is defined by a number n .
For example,

at the 1st term, $n = 1$

at the 2nd term, $n = 2$

at the 3rd term, $n = 3$

at the 4th term, $n = \boxed{4} \dots$

Given the sequence 1, 3, 5, 7, 9, 11, 13, ...

When $n = 1$, the term is 1, which can be expressed as n

When $n = 2$, the term is 3, which can be expressed as $n + 1$

When $n = 3$, the term is 5, which can be expressed as $n + 2$

When $n = 4$, the term is $\boxed{7}$, which can be expressed as $\boxed{n + 3}$

When $n = 5$, the term is $\boxed{9}$, which can be expressed as $\boxed{n + 4} \dots$

In general, each of the above terms can be expressed as $n + (n - 1)$,
which becomes $\boxed{2n} - 1 \dots \textcircled{1}$

Expression $\textcircled{1}$ is called the **general term**, or n^{th} term of the sequence.
By substituting the appropriate value of n into the general term, we can obtain any term in the sequence.

Answers: 4, 7, $n + 3$, 9, $n + 4$, $2n$

3. Use the general term, $\textcircled{1}$, to obtain the following terms.

(1) 10th term

[Sol] Substituting $n = 10$ into the general term, $\boxed{2(10)} - 1 = \boxed{19}$

Therefore, the 10th term is $\boxed{19}$.

(2) 21st term

[Sol] Substituting $n = 21$ into the general term, $2(21) - 1 = 41$

Therefore, the 21st term is 41.

(3) 52nd term

[Sol] Substituting $n = 52$ into the general term, $2(52) - 1 = 103$

Therefore, the 52nd term is 103.

(4) 100th term

[Sol] Substituting $n = 100$ into the general term, $2(100) - 1 = 199$

Therefore, the 100th term is 199.

Arithmetic Sequences

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	1	2-3	4-5	6-

1. In each of the following exercises, given the general term, obtain the 1st through 6th terms of the sequence.

(1) $n + 1$

2, 3, $\boxed{4}$, $\boxed{5}$, $\boxed{6}$, $\boxed{7}$

(2) $2n + 1$

3, 5, 7, 9, 11, 13

(3) $3n - 1$

2, 5, 8, 11, 14, 17

(4) $\frac{n}{n+1}$

 $\frac{1}{2}$, $\frac{2}{\boxed{3}}$, $\frac{\boxed{3}}{\boxed{4}}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$

(5) $\frac{2n-1}{2n}$

 $\frac{1}{\boxed{2}}$, $\frac{3}{\boxed{4}}$, $\frac{5}{6}$, $\frac{7}{8}$, $\frac{9}{10}$, $\frac{11}{12}$

(6) $3 - n$

2, 1, 0, -1, -2, -3

(7) $6 - 2n$

4, 2, 0, -2, -4, -6

(8) 2^n

2, $\boxed{4}$, 8, 16, 32, 64

(9) $(-1)^n$

-1, 1, -1, 1, -1, 1

(10) $\frac{n-1}{3}$

 $\boxed{0}$, $\frac{\boxed{1}}{\boxed{3}}$, $\frac{\boxed{2}}{\boxed{3}}$, $\boxed{1}$, $\frac{4}{3}$, $\frac{5}{3}$

Notation: Sequences can either be written by listing the terms as: $a_1, a_2, a_3, \dots, a_n, \dots$ or denoted with brace notation as $\{a_n\}$.
 a_n is used to represent the n^{th} term, i.e. the general term of a sequence.

N 32 b

2. For each given sequence, obtain the general term.

(1) 2, 4, 6, 8, 10, 12, ...

$$a_n = 2 \boxed{n}$$

(2) 4, 5, 6, 7, 8, 9, ...

$$a_n = n + 3$$

(3) 3, 2, 1, 0, -1, -2, ...

$$a_n = 4 - n$$

(4) $1, \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \frac{5}{9}, \frac{6}{11}, \dots$

$$a_n = \frac{n}{\boxed{2}n - \boxed{1}}$$

(5) -2, -3, -4, -5, -6, -7, ...

$$a_n = -n - 1$$

(6) -4, -8, -12, -16, -20, -24, ...

$$a_n = -4n$$

(7) 7, 4, 1, -2, -5, ...

$$a_n = -3n + 10$$

(8) 3, 9, 27, 81, 243, ...

$$a_n = 3^n$$

(9) $\frac{1}{4}, \frac{3}{7}, \frac{5}{10}, \frac{7}{13}, \frac{9}{16}, \dots$

$$a_n = \frac{2n - 1}{3n + 1}$$

Fill in each of the following boxes with the appropriate number for the given sequence:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, \boxed{29}, \boxed{31}, 37, \dots$$

This is the sequence of *prime numbers* in order.

There are some sequences, such as the sequence of prime numbers, whose general terms cannot be expressed as a function of the natural number, n .

In this learning focus, we will study only those sequences whose general terms can be defined with an equation.

Answers: 29, 31

Terminology: Natural numbers are the integers starting from and including 1. (1, 2, 3, 4, 5, ...) Natural numbers are also known as counting numbers.

Arithmetic Sequences

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	1	2	3	4

In the sequence 3, 4, 5, 6, 7, 8 ..., starting from the 1st term, we add 1 to each term to get the next term.

In the sequence 5, 7, 9, 11, 13, 15 ..., starting from the 1st term, we add 2 to each term to get the next term.

In the sequence 6, 4, 2, 0, -2, -4 ..., starting from the 1st term, we add -2 to each term to get the next term.

Sequences whose terms are obtained by successively adding a fixed number onto the 1st term are called *arithmetic sequences*, and that fixed number is called the *common difference*.

Answer: 2, -2

1. In each of the following exercises, obtain the 1st term and the common difference of the given arithmetic sequence.

Ex. 10, 13, 16, 19, 22, ...

[Sol] First term: 10

Common difference: 3



Starting from the 1st term, we add 3 to each term to get the next term.

(1) 6, 9, 12, 15, 18, 21, ...

First term: 6

Common difference: 3

(2) 7, 9, 11, 13, 15, 17, ...

First term: 7

Common difference: 2

(3) 3, 9, 15, 21, 27, 33, ...

First term: 3

Common difference: 6

(4) 8, 4, 0, -4, -8, -12, ...

First term: 8

Common difference: -4

(5) 3, 0, -3, -6, -9, -12, ...

First term: 3

Common difference: -3

(6) -6, -11, -16, -21, -26, -31, ...

First term: -6

Common difference: -5

N 33 b

The arithmetic sequence with first term a and common difference d has the following terms:

$$a_1 = a, \quad a_2 = a + d, \quad a_3 = a + 2d, \quad a_4 = a + 3d, \dots$$

The general term is expressed as:

$$a_n = a + (n - 1)d \quad \dots \textcircled{1}$$

2. Obtain the general term a_n of each arithmetic sequence.

(1) 1, 5, 9, 13, 17, 21, ...

$$a = \boxed{1}$$

$$d = \boxed{4}$$

$$a_n = \boxed{4n - 3}$$

(2) -12, -10, -8, -6, -4, -2, ...

$$a = \boxed{-12}$$

$$d = \boxed{2}$$

$$a_n = \boxed{2n - 14}$$

Ex.

Obtain the 1st term of the arithmetic sequence whose common difference is -3 and 11th term is -27. Then, express the general term, a_n .

[Sol] Letting a be the first term,

$$a_{11} = a + (11 - 1)(-3) = -27$$

$$\therefore a = 3$$



Use equation ①, where $n = 11$ and $d = -3$.

The 11th term is denoted as a_{11} .

Therefore, the 1st term is 3, and $a_n = -3n + 6$.

3. Obtain the 1st term of the arithmetic sequence whose common difference is -2 and 14th term is -22. Then, express the general term, a_n .

[Sol] Letting a be the first term,

$$a_{14} = a + (14 - 1)(-2) = -22$$

$$\therefore a = 4$$

Therefore, the 1st term is 4, and $a_n = -2n + 6$.

4. Obtain the 1st term of the arithmetic sequence whose common difference is 4 and 20th term is 34. Then, express the general term, a_n .

[Sol] Letting a be the first term,

$$a_{20} = a + (20 - 1)(4) = 34$$

$$\therefore a = -42$$

Therefore, the 1st term is -42, and $a_n = 4n - 46$.

Arithmetic Sequences

Time : to : Date Name

100%	90%	80%	70%	60%
(mistake) 0	-	1	-	2

Ex. Obtain the common difference of the arithmetic sequence whose 1st term is 20 and 15th term is 90.

[Sol] Letting d be the common difference,

$$a_{15} = 20 + (15 - 1)d = 90$$

$$d = 5$$

Therefore, the common difference is 5.

1. Obtain the common difference of the arithmetic sequence whose 1st term is 10 and 18th term is 44.

[Sol] Letting d be the common difference,

$$a_{18} = 10 + (18 - 1)d = 44$$

$$d = 2$$

Therefore, the common difference is 2.

2. Obtain the common difference of the arithmetic sequence whose 1st term is 6 and 15th term is 48.

[Sol] Letting d be the common difference,

$$a_{15} = 6 + (15 - 1)d = 48$$

$$d = 3$$

Therefore, the common difference is 3.

N 34 b

3. Obtain the common difference of the arithmetic sequence whose 1st term is -25 and 5th term is -5 .

[Sol] Letting d be the common difference,

$$a_5 = -25 + (5 - 1)d = -5$$

$$d = 5$$

Therefore, the common difference is 5.

4. Obtain the common difference of the arithmetic sequence whose 1st term is -17 and 20th term is 59.

[Sol] Letting d be the common difference,

$$a_{20} = -17 + (20 - 1)d = 59$$

$$d = 4$$

Therefore, the common difference is 4.

5. Obtain the common difference of the arithmetic sequence whose 1st term is 20 and 40th term is -97 .

[Sol] Letting d be the common difference,

$$a_{40} = 20 + (40 - 1)d = -97$$

$$d = -3$$

Therefore, the common difference is -3 .

Arithmetic Sequences

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	1	-	2-

1. Obtain the first term and common difference of the arithmetic sequence whose 7th term is 11 and 20th term is -41.

[Sol] Letting a be the first term and d be the common difference,

$$a_7 = a + 6d = 11 \quad \dots \textcircled{1}$$

$$a_{20} = a + 19d = -41 \quad \dots \textcircled{2}$$

From $\textcircled{2} - \textcircled{1}$,

$$13d = -52$$

$$d = -4$$

Substituting this value into $\textcircled{1}$, $a = 35$

2. Obtain the first term and common difference of the arithmetic sequence whose 6th term is -23 and 17th term is 32.

[Sol] Letting a be the first term and d be the common difference,

$$a_6 = a + 5d = -23 \quad \dots \textcircled{1}$$

$$a_{17} = a + 16d = 32 \quad \dots \textcircled{2}$$

From $\textcircled{2} - \textcircled{1}$,

$$11d = 55$$

$$d = 5$$

Substituting this value into $\textcircled{1}$, $a = -48$

3. Obtain the first term and common difference of the arithmetic sequence whose 10th term is 63 and 35th term is -12.

[Sol] Letting a be the first term and d be the common difference,

$$a_{10} = a + 9d = 63 \quad \dots \textcircled{1}$$

$$a_{35} = a + 34d = -12 \quad \dots \textcircled{2}$$

From $\textcircled{2} - \textcircled{1}$,

$$25d = -75$$

$$d = -3$$

Substituting this value into $\textcircled{1}$, $a = 90$

N 35 b

4. Obtain the 20th term of the arithmetic sequence whose 5th term is -3 and 14th term is 24.

[Sol] Letting a be the first term and d be the common difference,

$$a_5 = \boxed{a + 4d = -3} \quad \dots \textcircled{1}$$

$$a_{14} = \boxed{a + 13d = 24} \quad \dots \textcircled{2}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$a = \boxed{-15}, d = \boxed{3}$$

Therefore, the 20th term, a_{20} , is:

$$a_{20} = -15 + 19 \times 3 = 42$$

5. Obtain the 31st term of the arithmetic sequence whose 3rd term is -10 and 20th term is 41.

[Sol] Letting a be the first term and d be the common difference,

$$a_3 = a + 2d = -10 \quad \dots \textcircled{1}$$

$$a_{20} = a + 19d = 41 \quad \dots \textcircled{2}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$a = -16, d = 3$$

Therefore, the 31st term, a_{31} , is:

$$a_{31} = -16 + 30 \times 3 = 74$$

6. Obtain the 45th term of the arithmetic sequence whose 7th term is 15 and 30th term is -31 .

[Sol] Letting a be the first term and d be the common difference,

$$a_7 = a + 6d = 15 \quad \dots \textcircled{1}$$

$$a_{30} = a + 29d = -31 \quad \dots \textcircled{2}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$a = 27, d = -2$$

Therefore, the 45th term, a_{45} , is:

$$a_{45} = 27 + 44(-2) = -61$$

Arithmetic Sequences

Time : : to : : Date : : Name : : _____

100%	90%	80%	70%	60%
(mistakes) 0	—	1	—	2—

1. Obtain the 1st term of the arithmetic sequence whose common difference is -5 and 26th term is -30 .

[Sol] Letting a be the first term,

$$a_{26} = a + 25(-5) = -30$$

$$a = 95$$

Therefore, the 1st term is 95.

2. Obtain the common difference of the arithmetic sequence whose 1st term is 22 and 15th term is 50.

[Sol] Letting d be the common difference,

$$a_{15} = 22 + 14d = 50$$

$$d = 2$$

Therefore, the common difference is 2.

3. Obtain the first term and common difference of the arithmetic sequence whose 8th term is 16 and 19th term is -28 .

[Sol] Letting a be the first term and d be the common difference,

$$a_8 = a + 7d = 16 \quad \dots \textcircled{1}$$

$$a_{19} = a + 18d = -28 \quad \dots \textcircled{2}$$

From $\textcircled{2} - \textcircled{1}$,

$$11d = -44$$

$$d = -4$$

Substituting this value into $\textcircled{1}$,

$$a = 44$$

N 36 b

4. Obtain the 1st term of the arithmetic sequence whose common difference is 4 and 17th term is 8.

[Sol] Letting a be the first term,

$$a_{17} = a + 16 \cdot 4 = 8$$

$$a = -56$$

Therefore, the 1st term is -56 .

5. Obtain the common difference of the arithmetic sequence whose 1st term is 6 and 24th term is -63 .

[Sol] Letting d be the common difference,

$$a_{24} = 6 + 23d = -63$$

$$d = -3$$

Therefore, the common difference is -3 .

6. Obtain the 18th term of the arithmetic sequence whose 4th term is -4 and 12th term is 28.

[Sol] Letting a be the first term and d be the common difference,

$$a_4 = a + 3d = -4 \quad \dots \textcircled{1}$$

$$a_{12} = a + 11d = 28 \quad \dots \textcircled{2}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$a = -16, d = 4$$

Therefore, the 18th term, a_{18} , is:

$$a_{18} = -16 + 17 \times 4 = 52$$

Arithmetic Sequences

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	1	2	3	4

1. Obtain the first term, a , and common difference, d , of each arithmetic sequence whose general term, a_n , is given below:

(1) $a_n = 5n$

[Sol] Since this sequence is 5, 10, 15, 20,

First term: 5

Common difference: 5

(2) $a_n = 5n - 3$

[Sol] Since this sequence is 2, 7, 12, 17,

First term: 2

Common difference: 5

(3) $a_n = pn + q$

[Sol] Since this sequence is $p + q, 2p + q, 3p + q, \dots$

First term: $p + q$

Common difference: p

(4) $a_n = (p + 3)n + p + q$

[Sol] When $n = 1$, $(p + 3) \cdot 1 + p + q = 2p + q + 3$

When $n = 2$, $(p + 3) \cdot 2 + p + q = 3p + q + 6$

$\vdots$

Therefore,

First term: $2p + q + 3$

Common difference: $p + 3$

N 37 b

2. Given the sequence $\{a_n\}$ whose general term is $a_n = 2n + 1$, and the sequence $\{b_n\}$ whose general term is $b_n = 3n$, obtain the first term and common difference of the sequence $\{a_n - b_n\}$.

[Sol] The general term of $\{a_n - b_n\}$ is $(2n + 1) - \boxed{3n} = \boxed{-n + 1}$.

Therefore,

First term: $\boxed{0}$

Common difference: $\boxed{-1}$

3. Given the sequence $\{a_n\}$ whose general term is $a_n = 6n - 2$, and the sequence $\{b_n\}$ whose general term is $b_n = 2n + 6$, obtain the first term and common difference of the sequence $\{a_n - b_n\}$.

[Sol] The general term of $\{a_n - b_n\}$ is $(6n - 2) - (2n + 6) = 4n - 8$.

Therefore,

First term: -4

Common difference: 4

4. Given the sequence $\{a_n\}$ whose general term is $a_n = -8n - 3$, and the sequence $\{b_n\}$ whose general term is $b_n = 2n - 6$, obtain the first term and common difference of the sequence $\{a_n + b_n\}$.

[Sol] The general term of $\{a_n + b_n\}$ is $(-8n - 3) + (2n - 6) = -6n - 9$.

Therefore,

First term: -15

Common difference: -6

Arithmetic Sequences

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	1	2	3	4

1. Obtain the five numbers which when inserted between 7 and 25 cause all to become an arithmetic sequence.

[Sol] If the 1st term is 7, then the 7th term is 25.

Letting d be the common difference, and letting $a = 7$,

$$a_7 = 7 + 6d = 25$$

$$d = 3$$

Therefore, the general term a_n is:

$$a_n = 7 + (n-1)3 = 3n + 4 \quad \dots \textcircled{1}$$

Substituting 2, 3, 4, 5 and 6 into $\textcircled{1}$ for n ,

The five numbers are:

10, 13, 16, 19, 22

2. Obtain the six numbers which when inserted between -9 and 40 cause all to become an arithmetic sequence.

[Sol] Letting d be the common difference, and letting $a = -9$,

$$a_8 = -9 + 7d = 40$$

$$d = 7$$

Therefore, the general term a_n is:

$$a_n = -9 + (n-1)7 = 7n - 16 \quad \dots \textcircled{1}$$

Substituting 2, 3, 4, 5, 6 and 7 into $\textcircled{1}$ for n ,

The six numbers are:

$-2, 5, 12, 19, 26, 33$

N 38 b

3. Obtain the four numbers which when inserted between -20 and 10 cause all to become an arithmetic sequence.

[Sol] Letting d be the common difference, and letting $a = -20$,

$$a_6 = -20 + 5d = 10$$

$$d = 6$$

Therefore, the general term a_n is:

$$a_n = -20 + (n-1)6 = 6n - 26 \quad \dots \textcircled{1}$$

Substituting 2, 3, 4 and 5 into $\textcircled{1}$ for n ,

The four numbers are:

$$-14, -8, -2, 4$$

4. Obtain the five numbers which when inserted between 4 and 40 cause all to become an arithmetic sequence.

[Sol] Letting d be the common difference, and letting $a = 4$,

$$a_7 = 4 + 6d = 40$$

$$d = 6$$

Therefore, the general term a_n is:

$$a_n = 4 + (n-1)6 = 6n - 2 \quad \dots \textcircled{1}$$

Substituting 2, 3, 4, 5 and 6 into $\textcircled{1}$ for n ,

The five numbers are:

$$10, 16, 22, 28, 34$$

Arithmetic Sequences

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	—	—	—	—

1. Obtain the three numbers whose sum is 27 and product is 693, which form an arithmetic sequence.

[Sol] Letting $c - d$, c and $c + d$ be the three numbers,

$$\begin{cases} (c-d) + \boxed{c} + (\boxed{c+d}) = 27 & \dots \textcircled{1} \\ c(\boxed{c-d})(\boxed{c+d}) = 693 & \dots \textcircled{2} \end{cases}$$

Simplifying the LHS of each equation,

$$\begin{cases} \boxed{3c = 27} & \dots \textcircled{3} \\ \boxed{c(c^2 - d^2) = 693} & \dots \textcircled{4} \end{cases}$$

From $\textcircled{3}$, $c = \boxed{9}$

Substituting this value into $\textcircled{4}$,

$$9(81 - d^2) = 693$$

$$d = \boxed{\pm 2}$$

Therefore, the three numbers, $c - d$, c and $c + d$, are:

7, 9, 11

N 39 b

2. Obtain the five numbers whose sum is 30 and product is 3840, which form an arithmetic sequence.

[Sol] Letting $c - 2d, c - d, c, c + d$ and $c + 2d$ be the five numbers,

$$\begin{cases} (c - 2d) + (c - d) + c + (c + d) + (c + 2d) = 30 & \dots \textcircled{1} \\ c(c - 2d)(c - d)(c + d)(c + 2d) = 3840 & \dots \textcircled{2} \end{cases}$$

Simplifying the LHS of each equation,

$$\begin{cases} 5c = 30 & \dots \textcircled{3} \\ c(c^2 - d^2)(c^2 - 4d^2) = 3840 & \dots \textcircled{4} \end{cases}$$

From $\textcircled{3}$, $c = 6$

Substituting this value into $\textcircled{4}$,

$$(36 - d^2)(36 - 4d^2) = 640$$

$$(36 - d^2)(9 - d^2) = 160$$

$$164 - 45d^2 + d^4 = 0$$

$$(d^2 - 4)(d^2 - 41) = 0$$

$$d = \pm 2, \pm \sqrt{41}$$

Therefore, the five numbers, $c - 2d, c - d, c, c + d$ and $c + 2d$, are:

$$2, 4, 6, 8, 10 \quad \text{or} \quad 6 - 2\sqrt{41}, 6 - \sqrt{41}, 6, 6 + \sqrt{41}, 6 + 2\sqrt{41}$$

Arithmetic Sequences

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	1	-	2

1. Obtain the 1st term of the arithmetic sequence whose common difference is -4 and 12th term is -50 .

[Sol] Letting a be the first term,

$$a_{12} = a + (12 - 1)(-4) = -50$$

$$a = -6$$

Therefore, the 1st term is -6 .

2. Obtain the common difference of an arithmetic sequence whose 1st term is -12 and 22nd term is 93 .

[Sol] Letting d be the common difference,

$$a_{22} = -12 + (22 - 1)d = 93$$

$$d = 5$$

Therefore, the common difference is 5 .

3. Obtain the 30th term of an arithmetic sequence whose 5th term is -3 and 14th term is 24 .

[Sol] Letting a be the first term and d be the common difference,

$$a_5 = a + 4d = -3 \quad \dots \textcircled{1}$$

$$a_{14} = a + 13d = 24 \quad \dots \textcircled{2}$$

From $\textcircled{1}$ and $\textcircled{2}$, $d = 3$ and $a = -15$.

Therefore, $a_{30} = -15 + 29 \times 3 = 72$

N 40 b

4. Obtain the four numbers which when inserted between -5 and 25 cause all to become an arithmetic sequence.

[Sol] Letting d be the common difference, and letting $a = -5$,

$$a_6 = -5 + 5d = 25$$

$$d = 6$$

Therefore, the general term a_n is:

$$a_n = -5 + (n-1)6 = 6n - 11 \quad \dots \textcircled{1}$$

Substituting 2, 3, 4 and 5 into $\textcircled{1}$ for n ,

The four numbers are:

$$1, 7, 13, 19$$

5. Obtain the three numbers whose sum is 15 and product is 105, which form an arithmetic sequence.

[Sol] Letting $c-d$, c and $c+d$ be the three numbers,

$$\begin{cases} (c-d) + c + (c+d) = 15 & \dots \textcircled{1} \\ c(c-d)(c+d) = 105 & \dots \textcircled{2} \end{cases}$$

Simplifying the LHS of each equation,

$$\begin{cases} 3c = 15 & \dots \textcircled{3} \\ c(c^2 - d^2) = 105 & \dots \textcircled{4} \end{cases}$$

From $\textcircled{3}$, $c = 5$

Substituting this value into $\textcircled{4}$,

$$5(25 - d^2) = 105$$

$$d = \pm 2$$

Therefore, the three numbers, $c-d$, c and $c+d$, are:

$$3, 5, 7$$

Sums of Arithmetic Sequences

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	1	2	3	4

1. Given the arithmetic sequence 1, 4, 7, 10, 13, 16, ..., fill-in the blank boxes below.

- (1) The first term is . (2) The common difference is .

Expressing each term of the sequence in terms of the first term and the common difference,

		(3)	(4)	(5)	(6)	
1 st term	2 nd term	3 rd term	4 th term	5 th term	6 th term	...
1	4	7	10	13	16	
1	1 + 3	1 + 2 · 3	1 + 3 · 3	1 + 4 · 3	1 + 5 · 3	...

N 41 b

2. Letting a be the first term, d be the common difference and l be the last term, i.e. the n^{th} term, complete the table by expressing each term of an arithmetic sequence in terms of a , d and l :

(1)	(2)	(3)	(4)	(5)					
1 st term	2 nd term	3 rd term	4 th term	5 th term	6 th term	...	$n-2$ term (3 rd to last term)	$n-1$ term (2 nd to last term)	n^{th} term (last term)
a	a_2	$a_{\boxed{3}}$	$a_{\boxed{4}}$	$a_{\boxed{5}}$	$a_{\boxed{6}}$		$a_{n-\boxed{2}}$	a_{n-1}	a_n
a	$a + d$	$a + \boxed{2}d$	$a + \boxed{3}d$	$a + \boxed{4}d$	$a + \boxed{5}d$	...	$l - \boxed{2}d$	$l - d$	l

Sum of an Arithmetic Sequence

Given an arithmetic sequence with first term a and common difference d , letting l be the last term (the n^{th} term), the sum S_n of the terms from the first term a to the last term l , can be expressed as follows:

$$S_n = a + (a + d) + (a + 2d) + (a + \boxed{3}d) + \dots + (l - d) + l \quad \dots \textcircled{1}$$

Reversing the order of the terms of this sequence,

$$S_n = l + (l - d) + (l - 2d) + (l - \boxed{3}d) + \dots + (a + d) + a \quad \dots \textcircled{2}$$

From $\textcircled{1} + \textcircled{2}$,

$$\begin{aligned} 2S_n &= (a + l) + (a + l) + (a + l) + (\boxed{a} + \boxed{l}) + \dots + (a + l) + (a + l) \\ &= n(\boxed{a} + \boxed{l}) \end{aligned}$$

Therefore,

$$S_n = \frac{n(\boxed{a} + \boxed{l})}{\boxed{2}}$$

Sums of Arithmetic Sequences

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	1	2	3	4

Given that a is the first term, l is the last term, and n is the number of terms in the sequence, the sum of n terms, S_n , is expressed by the following formula.

Sum of an Arithmetic Sequence

Formula 1

$$S_n = \frac{n(a + l)}{2}$$

1. Using Formula 1, obtain the sum of each of the following arithmetic sequences.

Ex. A sequence with first term 4, last term 16 and number of terms 12.

$$[\text{Sol}] S_{12} = \frac{12(4 + 16)}{2} = 120$$

- (1) A sequence with first term 5, last term 21 and number of terms 14.

$$[\text{Sol}] S_{14} = \frac{14(5 + 21)}{2} = 182$$

- (2) A sequence with first term 2, last term 30 and number of terms 20.

$$[\text{Sol}] S_{20} = \frac{20(2 + 30)}{2} = 320$$

- (3) A sequence with first term -6 , last term 40 and number of terms 10.

$$[\text{Sol}] S_{10} = \frac{10(-6 + 40)}{2} = 170$$

- (4) A sequence with first term -9 , last term 43 and number of terms 6.

$$[\text{Sol}] S_6 = \frac{6(-9 + 43)}{2} = 102$$

- (5) A sequence with first term -20 , last term -2 and number of terms 8.

$$[\text{Sol}] S_8 = \frac{8(-20 - 2)}{2} = -88$$

- (6) A sequence with first term -15 , last term -1 and number of terms 31.

$$[\text{Sol}] S_{31} = \frac{31(-15 - 1)}{2} = -248$$

N 42 b

Substituting $l = a + (n - 1)d$, into $S_n = \frac{n(a + l)}{2}$,

Sum of an Arithmetic Sequence

Formula 2

$$S_n = \frac{n[2a + (n - 1)d]}{2}$$

2. Using Formula 2, obtain the sum of each of the following arithmetic sequences.

- (1) 5, 9, 13, 17, (through the 15th term)

[Sol] First term: $a = 5$

Common difference: $d = 4$

Number of terms: $n = 15$

$$S_{15} = \frac{15[2 \times 5 + (15 - 1) \times 4]}{2} = 495$$

- (2) 9, 3, -3, -9, (through the 11th term)

[Sol] First term: $a = 9$

Common difference: $d = -6$

Number of terms: $n = 11$

$$S_{11} = \frac{11[2 \times 9 + (11 - 1) \times (-6)]}{2} = -231$$

- (3) 1, 2, 3, 4, (through the 100th term)

[Sol] First term: $a = 1$

Common difference: $d = 1$

Number of terms: $n = 100$

$$S_{100} = \frac{100[2 \times 1 + (100 - 1) \times 1]}{2} = 5050$$

- (4) 1, 3, 5, 7, (through the n^{th} term)

[Sol] First term: $a = 1$

Common difference: $d = 2$

Number of terms: n

$$S_n = \frac{n[2 \times 1 + (n - 1) \times 2]}{2} = n^2$$

Sums of Arithmetic Sequences

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	1	2	3	4

$$S_n = \frac{n(a + l)}{2}$$

$$S_n = \frac{n[2a + (n - 1)d]}{2}$$

Obtain the sum of each of the following arithmetic sequences using either of the above formulas.

- (1) A sequence with 1st term 3, last term 21 and number of terms 8.

$$[\text{Sol}] S_8 = \frac{8(3 + 21)}{2} = 96$$

- (2) 6, 3, 0, -3, (through the 15th term)

$$[\text{Sol}] S_{15} = \frac{15[2 \times 6 + (15 - 1) \times (-3)]}{2} = -225$$

- (3) A sequence with 1st term 6 and 20th term 38, (through the 20th term).

$$[\text{Sol}] S_{20} = \frac{20(6 + 38)}{2} = 440$$

- (4) A sequence with 1st term 10 and 16th term 25, (through the 30th term).

[Sol] Letting d be the common difference,

$$a_{16} = 10 + 15d = 25$$

$$d = 1$$

$$S_{30} = \frac{30[2 \times 10 + (30 - 1) \times 1]}{2} = 735$$

- (5) A sequence with 1st term -4 and 13th term 32, (through the 24th term).

[Sol] Letting d be the common difference,

$$a_{13} = -4 + 12d = 32$$

$$d = 3$$

$$S_{24} = \frac{24[2 \times (-4) + (24 - 1) \times 3]}{2} = 732$$

N 43 b

- (6) A sequence with 1st term 50 and 8th term 29, (through the n^{th} term).

[Sol] Letting d be the common difference,

$$a_8 = 50 + 7d = 29$$

$$d = -3$$

$$S_n = \frac{n[2 \times 50 + (n-1) \times (-3)]}{2} = -\frac{n(3n-103)}{2}$$

- (7) A sequence where $a_8 = 9$ and $a_{13} = 24$, (through the 21st term).

[Sol] Letting a be the first term and d be the common difference,

$$a_8 = a + \boxed{7}d = \boxed{9} \quad \dots \textcircled{1}$$

$$a_{13} = a + \boxed{12}d = \boxed{24} \quad \dots \textcircled{2}$$

$$\text{From } \textcircled{1} \text{ and } \textcircled{2}, a = \boxed{-12} \text{ and } d = \boxed{3}.$$

$$S_{21} = \frac{21[2 \times (-12) + (21-1) \times 3]}{2} = 378$$

- (8) A sequence where $a_4 = -7$ and $a_{10} = 17$, (through the 15th term).

[Sol] Letting a be the first term and d be the common difference,

$$a_4 = a + 3d = -7 \quad \dots \textcircled{1}$$

$$a_{10} = a + 9d = 17 \quad \dots \textcircled{2}$$

$$\text{From } \textcircled{1} \text{ and } \textcircled{2}, a = -19 \text{ and } d = 4.$$

$$S_{15} = \frac{15[2 \times (-19) + (15-1) \times 4]}{2} = 135$$

- (9) A sequence where $a_3 = 5$ and $a_5 = 11$, (through the n^{th} term).

[Sol] Letting a be the first term and d be the common difference,

$$a_3 = a + 2d = 5 \quad \dots \textcircled{1}$$

$$a_5 = a + 4d = 11 \quad \dots \textcircled{2}$$

$$\text{From } \textcircled{1} \text{ and } \textcircled{2}, a = -1 \text{ and } d = 3.$$

$$S_n = \frac{n[2 \times (-1) + (n-1) \times 3]}{2} = \frac{n(3n-5)}{2}$$

Sums of Arithmetic Sequences

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	1	-	2

Letting a be the first term of an arithmetic sequence, d be the common difference, a_n be the n^{th} term and S_n be the sum of the first n terms, complete the following exercises.

- (1) Given $a = 16$, $d = -4$ and $S_n = -72$, obtain n .

$$[\text{Sol}] S_n = \frac{n[2 \times 16 + (n-1) \times (-4)]}{2} = -72$$

Simplifying, $(n-12)(n+3) = 0$

Since $n > 0$,

$$n = 12$$

- (2) Given $a = 10$, $a_n = -4$ and $S_n = 108$, obtain n and d .

$$[\text{Sol}] a_n = \boxed{10} + \boxed{(n-1)}d = \boxed{-4} \quad \dots \textcircled{1}$$

$$S_n = \frac{n[10 + \boxed{(-4)}]}{2} = \boxed{108} \quad \dots \textcircled{2}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$n = \boxed{36}, \quad d = \boxed{-\frac{2}{5}}$$

- (3) Given $a_5 = 28$ and $S_5 = 80$, obtain a and d .

$$[\text{Sol}] a_5 = a + 4d = 28 \quad \dots \textcircled{1}$$

$$S_5 = \frac{5(a+28)}{2} = 80 \quad \dots \textcircled{2}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$a = 4, d = 6$$

N 44 b

- (4) Given $a = 5, d = 2$ and $S_n = 60$, obtain n .

$$[\text{Sol}] S_n = \frac{n[2 \times 5 + (n-1) \times 2]}{2} = 60$$

Simplifying, $(n-6)(n+10) = 0$

Since $n > 0$,

$$n = 6$$

- (5) Given $a = -5, a_n = 6$ and $S_n = 50$, obtain n and d .

$$[\text{Sol}] a_n = -5 + (n-1)d = 6 \quad \dots \textcircled{1}$$

$$S_n = \frac{n[-5 + 6]}{2} = 50 \quad \dots \textcircled{2}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$n = 100, d = \frac{1}{9}$$

- (6) Given $a_8 = 12$ and $S_8 = 60$, obtain a and d .

$$[\text{Sol}] a_8 = a + 7d = 12 \quad \dots \textcircled{1}$$

$$S_8 = \frac{8(a + 12)}{2} = 60 \quad \dots \textcircled{2}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$a = 3, d = \frac{9}{7}$$

Sums of Arithmetic Sequences

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	1	-	2-

1. Given the arithmetic sequence 2, 9, 16, 23, ..., complete the following exercises.

- (1) Determine up through which numbered term the numbers of the sequence are less than 100.

[Sol] The general term is $\boxed{2} + \boxed{7}(n-1) = \boxed{7}n - \boxed{5}$.

$$\boxed{7}n - \boxed{5} < 100$$

$$n < \boxed{15}$$

Therefore, up through the $\boxed{14^{\text{th}}}$ term.

- (2) Obtain the sum of all the terms whose values are less than 100.

$$[\text{Sol}] S_{\boxed{14}} = \frac{14[2 \times 2 + (14-1) \times 7]}{2} = 665$$

2. Given the arithmetic sequence -5, -2, 1, 4, ..., complete the following exercises.

- (1) Determine up through which numbered term the numbers of the sequence are less than 85.

[Sol] The general term is $-5 + 3(n-1) = 3n - 8$.

$$3n - 8 < 85$$

$$n < 31$$

Therefore, up through the 30^{th} term.

- (2) Obtain the sum of all the terms whose values are less than 85.

$$[\text{Sol}] S_{30} = \frac{30[2 \times (-5) + (30-1) \times 3]}{2} = 1155$$

N 45 b

3. Given the arithmetic sequence 110, 116, 122, 128,, obtain the number of terms between 400 and 600, and their sum, S .

[Sol] Since the first term is 110 and the common difference is 6 , the general term, a_n , is:

$$a_n = 110 + 6(n - 1) = 6n + 104$$

$$400 < 6n + 104 < 600$$

$$49 \cdot 3 \dots\dots < n < 82 \cdot 6 \dots\dots$$

The number of terms from the 50^{th} term through the 82^{nd} term is:

$$82 - 50 + 1 = 33$$

Calculating their sum, S ,

$$a_{50} = 110 + 49 \times 6 = 404$$

$$a_{82} = 110 + 81 \times 6 = 596$$

Therefore,

$$S = \frac{33(404 + 596)}{2} = 16500$$

4. Given the arithmetic sequence $-6, -4, -2, 0, \dots\dots$, obtain the number of terms between 100 and 200, and their sum, S .

[Sol] Since the first term is -6 , and the common difference is 2 , the general term, a_n , is:

$$a_n = -6 + 2(n - 1) = 2n - 8$$

$$100 < 2n - 8 < 200$$

$$54 < n < 104$$

The number of terms from the 55^{th} term through the 103^{rd} term is:

$$103 - 55 + 1 = 49$$

Calculating their sum, S ,

$$a_{55} = -6 + 54 \times 2 = 102$$

$$a_{103} = -6 + 102 \times 2 = 198$$

$$\text{Therefore, } S = \frac{49(102 + 198)}{2} = 7350$$

Sums of Arithmetic Sequences

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	1	2	3	4

1. Obtain the sum of all the multiples of 7 between 200 and 300.

[Sol] $200 < 7n < 300$ (The multiples of 7 are denoted by $7n$.)

$$28.5 < n < 42.8$$

First term: $29 \times 7 = 203$

Last term: $42 \times 7 = 294$

Number of terms: $42 - 29 + 1 = 14$

Therefore,

$$S = \frac{14(203 + 294)}{2} = 3479$$

2. Obtain the sum of all the multiples of 12 between 400 and 600.

[Sol] $400 < 12n < 600$ (The multiples of 12 are denoted by $12n$.)

$$33.3 < n < 50$$

First term: $34 \times 12 = 408$

Last term: $49 \times 12 = 588$

Number of terms: $49 - 34 + 1 = 16$

Therefore,

$$S = \frac{16(408 + 588)}{2} = 7968$$

N 46 b

3. Obtain the total number of the three-digit integers which leave a remainder of 2 when divided by 7. Then, determine their sum, S .

[Sol] Any integer satisfying the given conditions can be expressed as:

$$\boxed{7}m + \boxed{2}$$

$$100 \leq \boxed{7}m + \boxed{2} \leq 999$$

$$\boxed{14} \leq m \leq \boxed{142.4} \dots\dots$$

$$\text{First term: } \boxed{14} \times 7 + 2 = \boxed{100}$$

$$\text{Last term: } \boxed{142} \times 7 + 2 = \boxed{996}$$

Number of terms: 129

Therefore,

$$S = \frac{129(100 + 996)}{2} = 70692$$

4. Obtain the total number of the four-digit integers which leave a remainder of 5 when divided by 9. Then, determine their sum, S .

[Sol] Any integer satisfying the given conditions can be expressed as:
 $9m + 5$

$$1000 \leq 9m + 5 \leq 9999$$

$$110.5\dots \leq m \leq 1110.4\dots\dots$$

$$\text{First term: } 111 \times 9 + 5 = 1004$$

$$\text{Last term: } 1110 \times 9 + 5 = 9995$$

Number of terms: 1000

Therefore,

$$S = \frac{1000(1004 + 9995)}{2} = 5,499,500$$

Sums of Arithmetic Sequences

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	1	-	2-

1. Given the set of all positive integers less than 100, solve the following exercises.

- (1) Obtain the sum of those divisible by 3.

[Sol] The first term is $\boxed{3}$, the last term is $\boxed{99}$,
and the number of terms is $\boxed{33}$.

Letting S_1 be the sum,

$$S_1 = \frac{33(3 + 99)}{2} = 1683$$

- (2) Obtain the sum of those divisible by 5.

[Sol] The first term is 5, the last term is 95,
and the number of terms is 19.

Letting S_2 be the sum,

$$S_2 = \frac{19(5 + 95)}{2} = 950$$

- (3) Obtain the sum of those divisible by both 3 and 5.

[Sol] Since these numbers are divisible by 15, the first term is $\boxed{15}$,
the last term is $\boxed{90}$, and the number of terms is $\boxed{6}$.

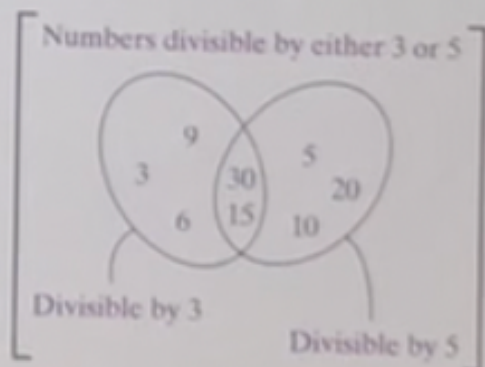
Letting S_3 be the sum,

$$S_3 = \frac{6(15 + 90)}{2} = 315$$

- (4) Obtain the sum of those divisible by either 3 or 5.

[Sol] Letting S be the sum,

$$\begin{aligned} S &= S_1 + \boxed{S_2} - \boxed{S_3} \\ &= 1683 + 950 - 315 \\ &= 2318 \end{aligned}$$



N 47 b

2. Obtain the sum of those integers from 100 to 200 (including both 100 and 200) which are divisible by either 3 or 5.

[Sol] Letting S_1, S_2 and S_3 be the sums of those numbers divisible by 3, 5 and 15,

$$S_1 = \frac{33(102 + 198)}{2} = 4950$$

$$S_2 = \frac{21(100 + 200)}{2} = 3150$$

$$S_3 = \frac{7(105 + 195)}{2} = 1050$$

Letting S be the sum in the problem,

$$S = S_1 + S_2 - S_3 = 7050$$

Sums of Arithmetic Sequences

Time : to : Date Name

100%	90%	80%	70%	60%~
(mistakes) 0	1	2-3	4	5-

The summation notation $\sum$ (sigma) is used to denote the sums of several values.

For example, the sum $a_3 + a_4 + a_5 + a_6$ is denoted by $\sum_{k=3}^6 a_k$.

(Note that other letters may be used in place of k .)

1. Letting $\{a_k\}$ be the sequence 1, 3, 5, 7, 9, ..., evaluate the following sums.

Ex.

$$\sum_{k=2}^4 a_k = a_2 + a_3 + a_4 = 3 + 5 + 7 = 15$$

$$(1) \sum_{k=3}^6 a_k = a_3 + a_4 + a_5 + a_6 = 5 + 7 + 9 + 11 = 32$$

$$(2) \sum_{k=1}^5 a_k = a_1 + a_2 + a_3 + a_4 + a_5 = 1 + 3 + 5 + 7 + 9 = 25$$

$$(3) \sum_{k=2}^2 a_k = a_2 = 3$$

$$(4) \sum_{k=1}^3 a_{k+1} = a_{1+1} + a_{2+1} + a_{3+1} = a_2 + a_3 + a_4 = 3 + 5 + 7 = 15$$

$$(5) \sum_{k=1}^3 a_{2k} = a_2 + a_4 + a_6 = 3 + 7 + 11 = 21$$

$$(6) \sum_{k=1}^2 a_{2k-1} = a_{2 \cdot 1 - 1} + a_{2 \cdot 2 - 1} = a_1 + a_3 = 1 + 5 = 6$$

N 48 b

2. Letting $\{a_k\}$ be the sequence $-9, -6, -3, 0, 3, \dots$, evaluate the following sums.

$$(1) \sum_{k=1}^4 a_k = a_1 + a_2 + a_3 + a_4 = -9 - 6 - 3 + 0 = -18$$

$$(2) \sum_{k=3}^5 a_k = a_3 + a_4 + a_5 = -3 + 0 + 3 = 0$$

$$(3) \sum_{k=2}^6 a_k = a_2 + a_3 + a_4 + a_5 + a_6 = -6 - 3 + 0 + 3 + 6 = 0$$

$$(4) \sum_{k=3}^3 a_k = a_3 = -3$$

$$(5) \sum_{k=5}^6 a_k = a_5 + a_6 = 3 + 6 = 9$$

$$(6) \sum_{k=1}^4 a_{k+1} = a_{1+1} + a_{2+1} + a_{3+1} + a_{4+1} \\ = a_2 + a_3 + a_4 + a_5 = -6 - 3 + 0 + 3 = -6$$

$$(7) \sum_{k=3}^5 a_{k-1} = a_{3-1} + a_{4-1} + a_{5-1} = a_2 + a_3 + a_4 = -6 - 3 + 0 = -9$$

$$(8) \sum_{k=5}^7 a_{k-2} = a_{5-2} + a_{6-2} + a_{7-2} = a_3 + a_4 + a_5 = -3 + 0 + 3 = 0$$

$$(9) \sum_{k=1}^3 a_{3k} = a_3 + a_6 + a_9 = -3 + 6 + 9 = 12$$

$$(10) \sum_{k=2}^4 a_{2k-1} = a_{2 \cdot 2 - 1} + a_{2 \cdot 3 - 1} + a_{2 \cdot 4 - 1} = a_3 + a_5 + a_7 = -3 + 3 + 9 = 9$$

Sums of Arithmetic Sequences

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	1	2-3	4-5	6-

1. Evaluate the following sums.

$$(1) \sum_{k=1}^5 (2k-1) = (2 \cdot 1 - 1) + (2 \cdot 2 - 1) + (2 \cdot 3 - 1) + (2 \cdot 4 - 1) + (2 \cdot 5 - 1) \\ = 1 + 3 + 5 + 7 + 9 = 25$$

$$(2) \sum_{k=1}^3 (k+n) = (1+n) + (2+\boxed{n}) + (\boxed{3+n}) = 3n+6$$

$$(3) \sum_{k=1}^3 (k-n) = (1-n) + (2-n) + (3-n) = -3n+6$$

$$(4) \sum_{n=1}^3 (k-n) = (k-\boxed{1}) + (\boxed{k-2}) + (\boxed{k-3}) = 3k-6$$

$$(5) \sum_{k=1}^n k = 1 + 2 + \boxed{3} + \cdots + n = \frac{n(1+\boxed{n})}{2}$$

$$(6) \sum_{k=1}^n (k+n) = (1+n) + (\boxed{2}+n) + (\boxed{3}+n) + \cdots + (n+n) \\ = (1+2+\cdots+n) + \boxed{n}^2 = \frac{3n^2+n}{2}$$

$$(7) \sum_{k=1}^n nk = n + 2n + 3n + \cdots + n^2 = n(1+2+3+\cdots+n) \\ = n \cdot \frac{n(1+n)}{2} = \frac{n^3+n^2}{2}$$

$$(8) \sum_{k=1}^n (n-k) = (n-1) + (n-2) + \cdots + [n-(n-1)] + (n-n) \\ = n^2 - (1+2+\cdots+n) \\ = n^2 - \frac{n(1+n)}{2} = \frac{n^2-n}{2}$$

N 49 b

2. Rewrite each of the following sums using sigma notation.

$$(1) \quad 1 + 3 + 5 + \dots + (2n - 1) = \sum_{k=1}^n (2k - 1)$$

$$(2) \quad 1 + 4 + 7 + \dots + (3n - 2) = \sum_{k=1}^n (3k - 2)$$

$$(3) \quad 2 + 5 + 8 + \dots + (n^{\text{th}} \text{ term}) = \sum_{k=1}^n (3k - 1)$$

$$(4) \quad 2 + 6 + 10 + \dots + (n^{\text{th}} \text{ term}) = \sum_{k=1}^n (4k - 2)$$

$$(5) \quad 3 + 5 + 7 + \dots + (n^{\text{th}} \text{ term}) = \sum_{k=1}^n (2k + 1)$$

$$(6) \quad 12 + 15 + 18 + \dots + (n^{\text{th}} \text{ term}) = \sum_{k=1}^n (3k + 9)$$

$$(7) \quad -5 - 2 + 1 + \dots + (n^{\text{th}} \text{ term}) = \sum_{k=1}^n (3k - 8)$$

$$(8) \quad -2 + 3 + 8 + \dots + (n^{\text{th}} \text{ term}) = \sum_{k=1}^n (5k - 7)$$

$$(9) \quad -10 - 5 + 0 + \dots + (n^{\text{th}} \text{ term}) = \sum_{k=1}^n (5k - 15)$$

$$(10) \quad -40 - 30 - 20 - \dots + (n^{\text{th}} \text{ term}) = \sum_{k=1}^n (10k - 50)$$

Sums of Arithmetic Sequences

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	1	2	3	4

1. Obtain the sum of each of the following arithmetic sequences.

(1) 3, 6, 9, 12, (through the 20th term)

[Sol] First term: $a = 3$

Common difference: $d = 3$

Number of terms: $n = 20$

$$S_{20} = \frac{20[2 \times 3 + (20 - 1) \times 3]}{2} = 630$$

(2) -15, -10, -5, 0, (through the 12th term)

[Sol] First term: $a = -15$

Common difference: $d = 5$

Number of terms: $n = 12$

$$S_{12} = \frac{12[2 \times (-15) + (12 - 1) \times 5]}{2} = 150$$

2. Let S_n be the sum of the first n terms of an arithmetic sequence.
Given $S_{10} = 100$ and $S_{20} = 400$, obtain S_{30} .

[Sol] Letting a be the first term and d be the common difference,

$$\frac{10(2a + 9d)}{2} = 100 \quad \dots \textcircled{1}$$

$$\frac{20(2a + 19d)}{2} = 400 \quad \dots \textcircled{2}$$

From $\textcircled{1}$ and $\textcircled{2}$, $a = 1$ and $d = 2$.

Therefore,

$$S_{30} = \frac{30(2 \times 1 + 29 \times 2)}{2} = 900$$

N 50 b

3. Assume that the sum of the first $n+2$ terms of the arithmetic sequence $-5, a_1, a_2, \dots, a_n, 15$, is 100, and obtain n .

$$[\text{Sol}] S_{n+2} = \frac{(n+2)(-5+15)}{2} = 5n+10$$

$$S_{n+2} = 100$$

$$5n+10 = 100$$

$$n = 18$$

4. Letting $\{a_k\}$ be the sequence $-3, -1, 1, 3, 5, \dots$, evaluate the following sums.

$$(1) \sum_{k=1}^5 a_k = a_1 + a_2 + a_3 + a_4 + a_5 = -3 - 1 + 1 + 3 + 5 = 5$$

$$(2) \sum_{k=1}^3 a_{k+1} = a_2 + a_3 + a_4 = -1 + 1 + 3 = 3$$

$$(3) \sum_{k=2}^4 a_{2k-1} = a_3 + a_5 + a_7 = 1 + 5 + 9 = 15$$

5. Evaluate the following sums.

$$\begin{aligned}(1) \sum_{k=1}^4 (3k-2) &= (3 \cdot 1 - 2) + (3 \cdot 2 - 2) + (3 \cdot 3 - 2) + (3 \cdot 4 - 2) \\ &= 1 + 4 + 7 + 10 \\ &= 22\end{aligned}$$

$$\begin{aligned}(2) \sum_{k=3}^7 (k+n) &= (3+n) + (4+n) + (5+n) + (6+n) + (7+n) \\ &= 5n + 25\end{aligned}$$

Geometric Sequences

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	1	2	3	4

In the sequence 1, 2, 4, 8, 16, starting from the 1st term, we multiply each term by 2 to get the next term.

In the sequence 1, 3, 9, 27, 81, starting from the 1st term, we multiply each term by 3 to get the next term.

In the sequence -4, 8, -16, 32, -64, starting from the 1st term, we multiply each term by -2 to get the next term.

Sequences whose terms are obtained by successively multiplying a fixed non-zero number with the 1st term, are called *geometric sequences*, and that fixed number is called the *common ratio*.

Answers: 3, -2

1. In each of the following exercises, obtain the 1st term and common ratio of the given geometric sequence.

(1) 1, 5, 25, 125, 625, ...

First term: 1

Common ratio: 5

(5) -5, 5, -5, 5, -5, ...

First term: -5

Common ratio: -1

(2) 2, 6, 18, 54, 162, ...

First term: 2

Common ratio: 3

(6) 4, -2, 1, $-\frac{1}{2}$, $\frac{1}{4}$, ...

First term: 4

Common ratio: $-\frac{1}{2}$

(3) -3, -6, -12, -24, -48, ...

First term: -3

Common ratio: 2

(7) $-\frac{1}{10}$, -1, -10, -100, -1000, ...

First term: $-\frac{1}{10}$

Common ratio: 10

(4) 8, 4, 2, 1, $\frac{1}{2}$, ...

First term: 8

Common ratio: $\frac{1}{2}$

(8) 6, 4, $\frac{8}{3}$, $\frac{16}{9}$, $\frac{32}{27}$, ...

First term: 6

Common ratio: $\frac{2}{3}$

N 51 b

The geometric sequence with the first term a and common ratio r has the following terms:

$$a_1 = a, \quad a_2 = ar, \quad a_3 = ar^2, \quad a_4 = ar^3, \dots$$

The general term a_n is expressed as:

$$a_n = ar^{n-1}$$

2. For each given geometric sequence, obtain the 1st term, a , the common ratio, r , and the general term, a_n .

(1) 4, 8, 16, 32, 64, ...

[Sol] $a = \boxed{4}, \quad r = \boxed{2},$

$$a_n = \boxed{4} \cdot \boxed{2}^{n-1} = 2^{\boxed{2}} \cdot \boxed{2}^{n-1} = 2^{\boxed{n+1}}$$

(2) -3, -9, -27, -81, -243, ...

[Sol] $a = -3, \quad r = 3,$

$$a_n = -3 \cdot 3^{n-1} = -3^n$$

(3) 16, 8, 4, 2, 1, ...

[Sol] $a = 16, \quad r = \frac{1}{2},$

$$a_n = 16 \cdot \left(\frac{1}{2}\right)^{n-1} = 2^4 \cdot \left(\frac{1}{2}\right)^{n-1} = \left(\frac{1}{2}\right)^{-4} \cdot \left(\frac{1}{2}\right)^{n-1} = \left(\frac{1}{2}\right)^{n-5}$$

4) -2, -8, -32, -128, -512, ...

[Sol] $a = -2, \quad r = 4,$

$$a_n = -2 \cdot 4^{n-1} = -2 \cdot 2^{2n-2} = -2^{2n-1}$$

5) $\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \frac{1}{54}, \frac{1}{162}, \dots$

[Sol] $a = \boxed{\frac{1}{2}}, \quad r = \boxed{\frac{1}{3}},$

$$a_n = \boxed{\frac{1}{2}} \cdot \boxed{\left(\frac{1}{3}\right)^{n-1}}$$

Geometric Sequences

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	1	2	3	4

For each given geometric sequence, obtain the 1st term, a , the common ratio, r , and the general term, a_n .

(1) 3, 6, 12, 24, 48, ...

[Sol] $a = 3, r = 2,$

$$a_n = 3 \cdot 2^{n-1}$$

(2) -2, -6, -18, -54, -162, ...

[Sol] $a = -2, r = 3,$

$$a_n = -2 \cdot 3^{n-1}$$

(3) 36, 12, 4, $\frac{4}{3}, \frac{4}{9}, \dots$

[Sol] $a = 36, r = \frac{1}{3},$

$$a_n = 36 \cdot \left(\frac{1}{3}\right)^{n-1} = 4 \cdot 3^2 \cdot \left(\frac{1}{3}\right)^{n-1} = 4 \cdot \left(\frac{1}{3}\right)^{n-3}$$

(4) -5, 10, -20, 40, -80, ...

[Sol] $a = -5, r = -2,$

$$a_n = -5 \cdot (-2)^{n-1}$$

(5) $\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \frac{1}{24}, \frac{1}{48}, \dots$

[Sol] $a = \frac{1}{3}, r = \frac{1}{2},$

$$a_n = \frac{1}{3} \cdot \left(\frac{1}{2}\right)^{n-1}$$

N 52 b

(6) $6, -6, 6, -6, 6, \dots$

[Sol] $a = 6, \quad r = -1,$

$$a_n = 6 \cdot (-1)^{n-1} \text{ [or } -6 \cdot (-1)^n]$$

(7) $24, -12, 6, -3, \frac{3}{2}, \dots$

[Sol] $a = 24, \quad r = -\frac{1}{2},$

$$a_n = 24 \cdot \left(-\frac{1}{2}\right)^{n-1} = 3 \cdot 2^3 \cdot \left(-\frac{1}{2}\right)^{n-1} = -3 \cdot \left(-\frac{1}{2}\right)^{n-4}$$

(8) $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, \dots$

[Sol] $a = \frac{1}{8}, \quad r = 2,$

$$a_n = \frac{1}{8} \cdot 2^{n-1} = 2^{-3} \cdot 2^{n-1} = 2^{n-4}$$

(9) $\frac{5}{4}, \frac{5}{6}, \frac{5}{9}, \frac{10}{27}, \frac{20}{81}, \dots$

[Sol] $a = \frac{5}{4}, \quad r = \frac{2}{3},$

$$a_n = \frac{5}{4} \cdot \left(\frac{2}{3}\right)^{n-1}$$

(10) $25, -5, 1, -\frac{1}{5}, \frac{1}{25}, \dots$

[Sol] $a = 25, \quad r = -\frac{1}{5},$

$$a_n = 25 \cdot \left(-\frac{1}{5}\right)^{n-1} = \left(-\frac{1}{5}\right)^{n-3}$$

Geometric Sequences

Time : to : Date Name

100%	90%	80%	70%	60%~
(mistakes) 0	-	1	-	2-

- Ex. Obtain the 1st term of the geometric sequence whose common ratio is 2 and 4th term is 24.

[Sol] Letting a be the first term,

$$a_4 = a \times 2^3 = 24$$

$$\therefore a = 3$$

Therefore, the 1st term is 3.

1. Obtain the 1st term of the geometric sequence whose common ratio is -2 and 5th term is 64.

[Sol] Letting a be the first term,

$$a_5 = a \times (-2)^4 = 64$$

$$\therefore a = 4$$

Therefore, the 1st term is 4.

2. Obtain the 1st term of the geometric sequence whose common ratio is $\frac{1}{2}$ and 8th term is $\frac{1}{4}$.

[Sol] Letting a be the first term,

$$a_8 = a \times \left(\frac{1}{2}\right)^7 = \frac{1}{4}$$

$$\therefore a = 32$$

Therefore, the 1st term is 32.

3. Obtain the 1st term of the geometric sequence whose common ratio is 3 and 6th term is 27.

[Sol] Letting a be the first term,

$$a_6 = a \times 3^5 = 27$$

$$\therefore a = \frac{3^3}{3^5} = \frac{1}{9}$$

Therefore, the 1st term is $\frac{1}{9}$.

N 53 b

4. Obtain the 1st term of the geometric sequence whose common ratio is $\frac{2}{3}$ and 4th term is 32.

[Sol] Letting a be the first term,

$$a_4 = a \times \left(\frac{2}{3}\right)^3 = 32$$
$$\therefore a = 108$$

Therefore, the 1st term is 108.

5. Obtain the 1st term of the geometric sequence whose common ratio is $-\frac{1}{3}$ and 6th term is $\frac{1}{81}$.

[Sol] Letting a be the first term,

$$a_6 = a \times \left(-\frac{1}{3}\right)^5 = \frac{1}{81}$$
$$\therefore a = -3$$

Therefore, the 1st term is -3 .

6. Obtain the 1st term of the geometric sequence whose common ratio is -2 and 7th term is 48.

[Sol] Letting a be the first term,

$$a_7 = a \times (-2)^6 = 48$$
$$\therefore a = \frac{2^6 \cdot 3}{2^6} = \frac{3}{4}$$

Therefore, the 1st term is $\frac{3}{4}$.

Geometric Sequences

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	1	2	3	4

1. Obtain the common ratio of the geometric sequence whose 1st term is 5 and 5th term is 405.

[Sol] Letting r be the common ratio,

$$a_5 = 5r^4 = 405$$

$$r^4 = 81$$

$$\therefore r = \pm 3$$

2. Obtain the common ratio of the geometric sequence whose 1st term is 4 and 8th term is 512.

[Sol] Letting r be the common ratio,

$$a_8 = 4r^7 = 512$$

$$r^7 = 128$$

$$\therefore r = 2$$

3. Obtain the common ratio of the geometric sequence whose 1st term is -3 and 6th term is 729.

[Sol] Letting r be the common ratio,

$$a_6 = -3r^5 = 729$$

$$r^5 = -243$$

$$\therefore r = -3$$

N 54 b

4. Obtain the common ratio of the geometric sequence whose 1st term is 64 and 10th term is $-\frac{1}{8}$.

[Sol] Letting r be the common ratio,

$$a_{10} = 64r^9 = -\frac{1}{8}$$

$$r^9 = -\frac{1}{512}$$

$$\therefore r = -\frac{1}{2}$$

5. Obtain the common ratio of the geometric sequence whose 1st term is $4\frac{1}{6}$ and 5th term is $\frac{8}{75}$.

[Sol] Letting r be the common ratio,

$$a_5 = \frac{25}{6}r^4 = \frac{8}{75}$$

$$r^4 = \frac{16}{625}$$

$$\therefore r = \pm \frac{2}{5}$$

6. Obtain the common ratio of the geometric sequence whose 1st term is $2\frac{13}{16}$ and 4th term is $-6\frac{2}{3}$.

[Sol] Letting r be the common ratio,

$$a_4 = \frac{45}{16}r^3 = -\frac{20}{3}$$

$$r^3 = -\frac{64}{27}$$

$$\therefore r = -\frac{4}{3}$$

Geometric Sequences

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	—	—	1	2

1. Obtain the 1st term and common ratio of a geometric sequence whose 2nd term is -24 and 5th term is 81 .

[Sol] Letting a be the first term and r be the common ratio,

$$a_2 = ar = -24 \quad \dots \textcircled{1}$$

$$a_5 = ar^4 = 81 \quad \dots \textcircled{2}$$

From $\textcircled{2} \div \textcircled{1}$,

$$r^3 = \frac{27}{-8}$$

$$\therefore r = -\frac{3}{2}$$

Substituting this value into $\textcircled{1}$,

$$a = 16$$

2. Obtain the 1st term and common ratio of a geometric sequence whose 6th term is 25 and 8th term is 100 .

[Sol] Letting a be the first term and r be the common ratio,

$$a_6 = ar^5 = 25 \quad \dots \textcircled{1}$$

$$a_8 = ar^7 = 100 \quad \dots \textcircled{2}$$

From $\textcircled{2} \div \textcircled{1}$,

$$r^2 = 4$$

$$\therefore r = \pm 2$$

Substituting this value into $\textcircled{1}$,

$$a = \pm \frac{25}{32}$$

N 55 b

3. Obtain the 1st term and common ratio of a geometric sequence whose 4th term is 0.5 and 6th term is 12.5.

[Sol] Letting a be the first term and r be the common ratio,

$$a_4 = ar^3 = \frac{1}{2} \quad \dots \textcircled{1}$$

$$a_6 = ar^5 = \frac{25}{2} \quad \dots \textcircled{2}$$

From $\textcircled{2} \div \textcircled{1}$,

$$r^2 = 25$$

$$\therefore r = \pm 5$$

Substituting this value into $\textcircled{1}$,

$$a = \pm \frac{1}{250}$$

4. Obtain the 1st term and common ratio of a geometric sequence whose 5th term is $\frac{1}{64}$ and 7th term is $\frac{1}{16}$.

[Sol] Letting a be the first term and r be the common ratio,

$$a_5 = ar^4 = \frac{1}{64} \quad \dots \textcircled{1}$$

$$a_7 = ar^6 = \frac{1}{16} \quad \dots \textcircled{2}$$

From $\textcircled{2} \div \textcircled{1}$,

$$r^2 = 4$$

$$\therefore r = \pm 2$$

Substituting this value into $\textcircled{1}$,

$$a = \frac{1}{1024}$$

Geometric Sequences

Time : to : Date Name

100%	90%	80%	70%	60%~
(mistakes) 0	-	-	1	2-

1. Obtain the general term of a geometric sequence whose 3rd term is 2 and 6th term is $\frac{1}{4}$. Assume each term of the sequence is a real number.

[Sol] Letting a be the first term and r be the common ratio,

$$a_3 = ar^2 = 2 \quad \dots \textcircled{1}$$

$$a_6 = ar^5 = \frac{1}{4} \quad \dots \textcircled{2}$$

From $\textcircled{2} \div \textcircled{1}$,

$$r^3 = \frac{1}{8}$$

$$\therefore r = \frac{1}{2}$$

Substituting this value into $\textcircled{1}$,

$$a = 8$$

$$\therefore a_n = 8 \cdot \left(\frac{1}{2}\right)^{n-1} = \left(\frac{1}{2}\right)^{n-4}$$

2. Obtain the general term of a geometric sequence whose 2nd term is $\frac{1}{6}$ and 4th term is 6. Assume each term of the sequence is a real number.

[Sol] Letting a be the first term and r be the common ratio,

$$a_2 = ar = \frac{1}{6} \quad \dots \textcircled{1}$$

$$a_4 = ar^3 = 6 \quad \dots \textcircled{2}$$

From $\textcircled{2} \div \textcircled{1}$,

$$r^2 = 36$$

$$\therefore r = \pm 6$$

Substituting this value into $\textcircled{1}$,

$$a = \pm \frac{1}{36}$$

$$\text{When } r = 6 \text{ and } a = \frac{1}{36}, \quad a_n = \frac{1}{36} \cdot 6^{n-1} = 6^{-2} \cdot 6^{n-1} = 6^{n-3}$$

$$\begin{aligned} \text{When } r = -6 \text{ and } a = -\frac{1}{36}, \quad a_n &= -\frac{1}{36} \cdot (-6)^{n-1} = -6^{-2} \cdot (-1)^{n-1} \cdot 6^{n-1} \\ &= (-1)^n \cdot 6^{n-3} \end{aligned}$$

N 56 b

3. Obtain the general term of a geometric sequence whose 3rd term is 10 and 8th term is $\frac{5}{16}$. Assume each term of the sequence is a real number.

[Sol] Letting a be the first term and r be the common ratio,

$$a_3 = ar^2 = 10 \quad \dots \textcircled{1}$$

$$a_8 = ar^7 = \frac{5}{16} \quad \dots \textcircled{2}$$

From $\textcircled{2} \div \textcircled{1}$,

$$r^5 = \frac{1}{32}$$

$$\therefore r = \frac{1}{2}$$

Substituting this value into $\textcircled{1}$,

$$a = 40$$

$$\therefore a_n = 40 \cdot \left(\frac{1}{2}\right)^{n-1} \quad \left[= 5 \cdot \left(\frac{1}{2}\right)^{n-4} \right]$$

4. Obtain the general term of a geometric sequence whose 2nd term is $\frac{9}{10}$ and 5th term is $-\frac{25}{48}$. Assume each term of the sequence is a real number.

[Sol] Letting a be the first term and r be the common ratio,

$$a_2 = ar = \frac{9}{10} \quad \dots \textcircled{1}$$

$$a_5 = ar^4 = -\frac{25}{48} \quad \dots \textcircled{2}$$

From $\textcircled{2} \div \textcircled{1}$,

$$r^3 = -\frac{125}{216}$$

$$\therefore r = -\frac{5}{6}$$

Substituting this value into $\textcircled{1}$,

$$a = -\frac{27}{25}$$

$$\therefore a_n = -\frac{27}{25} \cdot \left(-\frac{5}{6}\right)^{n-1}$$

Geometric Sequences

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	-	1	2-

1. Obtain the three real numbers, whose sum is 7 and product is 8, which form a geometric sequence.

[Sol] Letting r be the common ratio; and a , ar and ar^2 be the three real numbers forming the geometric sequence,

$$\begin{aligned} a + ar + ar^2 &= 7 & a \cdot ar \cdot ar^2 &= 8 \\ \therefore a(1 + r + r^2) &= 7 & ar &= 2 \end{aligned} \quad \dots \textcircled{1} \quad \dots \textcircled{2}$$

From ① and ②,

$$\begin{aligned} 2r^2 - 5r + 2 &= 0 \\ \therefore r &= \frac{1}{2}, 2 \\ \therefore a &= 4, 1 \end{aligned}$$

When $a = 4$ and $r = \frac{1}{2}$, the numbers are: 4, 2 and 1

When $a = 1$ and $r = 2$, the numbers are: 1, 2 and 4

Therefore, the three real numbers are 1, 2 and 4.

2. Obtain the three real numbers, whose sum is -7 and product is 27, which form a geometric sequence.

[Sol] Letting r be the common ratio; and a , ar and ar^2 be the three real numbers forming the geometric sequence,

$$\begin{aligned} a + ar + ar^2 &= -7 & a \cdot ar \cdot ar^2 &= 27 \\ \therefore a(1 + r + r^2) &= -7 & ar &= 3 \end{aligned} \quad \dots \textcircled{1} \quad \dots \textcircled{2}$$

From ① and ②,

$$\begin{aligned} 3r^2 + 10r + 3 &= 0 \\ \therefore r &= -\frac{1}{3}, -3 \\ \therefore a &= -9, -1 \end{aligned}$$

When $a = -9$ and $r = -\frac{1}{3}$, the numbers are: $-9, 3$ and -1

When $a = -1$ and $r = -3$, the numbers are: $-1, 3$ and -9

Therefore, the three real numbers are $-9, -1$ and 3 .

N 57 b

3. Obtain the three real numbers, whose sum is $2\frac{1}{6}$ and product is $\frac{1}{8}$, which form a geometric sequence.

[Sol] Letting r be the common ratio; and a , ar and ar^2 be the three real numbers forming the geometric sequence,

$$a + ar + ar^2 = 2\frac{1}{6} \qquad a \cdot ar \cdot ar^2 = \frac{1}{8}$$

$$\therefore a(1 + r + r^2) = \frac{13}{6} \qquad \dots \textcircled{1} \qquad ar = \frac{1}{2} \qquad \dots \textcircled{2}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$3r^2 - 10r + 3 = 0$$

$$\therefore r = \frac{1}{3}, 3$$

$$\therefore a = \frac{3}{2}, \frac{1}{6}$$

When $a = \frac{3}{2}$ and $r = \frac{1}{3}$, the numbers are: $\frac{3}{2}, \frac{1}{2}$ and $\frac{1}{6}$

When $a = \frac{1}{6}$ and $r = 3$, the numbers are: $\frac{1}{6}, \frac{1}{2}$ and $\frac{3}{2}$

Therefore, the three real numbers are $\frac{1}{6}, \frac{1}{2}$ and $\frac{3}{2}$.

4. Obtain the three real numbers, whose sum is $4\frac{2}{3}$ and product is -64 , which form a geometric sequence.

[Sol] Letting r be the common ratio; and a , ar and ar^2 be the three real numbers forming the geometric sequence,

$$a + ar + ar^2 = 4\frac{2}{3} \qquad a \cdot ar \cdot ar^2 = -64$$

$$\therefore a(1 + r + r^2) = \frac{14}{3} \qquad \dots \textcircled{1} \qquad ar = -4 \qquad \dots \textcircled{2}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$6r^2 + 13r + 6 = 0$$

$$\therefore r = -\frac{3}{2}, -\frac{2}{3}$$

$$\therefore a = \frac{8}{3}, 6$$

When $a = \frac{8}{3}$ and $r = -\frac{3}{2}$, the numbers are: $\frac{8}{3}, -4$ and 6

When $a = 6$ and $r = -\frac{2}{3}$, the numbers are: $6, -4$ and $\frac{8}{3}$

Therefore, the three real numbers are $-4, \frac{8}{3}$ and 6 .

N 58 a

Geometric Sequences

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	1	2	3	4

1. For each given geometric sequence, obtain the 1st term, a , the common ratio r , and the general term, a_n .

(1) 1, 4, 16, 64, 256, ...

[Sol] $a = 1, \quad r = 4,$

$$a_n = 1 \cdot 4^{n-1} = 4^{n-1}$$

(2) -3, 6, -12, 24, -48, ...

[Sol] $a = -3, \quad r = -2,$

$$a_n = -3 \cdot (-2)^{n-1}$$

(3) $\frac{1}{27}, \frac{1}{9}, \frac{1}{3}, 1, 3, \dots$

[Sol] $a = \frac{1}{27}, \quad r = 3,$

$$a_n = \frac{1}{27} \cdot 3^{n-1} = 3^{-3} \cdot 3^{n-1} = 3^{n-4}$$

(4) 3, -1, $\frac{1}{3}, -\frac{1}{9}, \frac{1}{27}, \dots$

[Sol] $a = 3, \quad r = -\frac{1}{3}$

$$a_n = 3 \cdot \left(-\frac{1}{3}\right)^{n-1} = -\left(-\frac{1}{3}\right)^{n-2}$$

2. Obtain the 1st term of the geometric sequence whose common ratio is -3 and 5th term is 9.

[Sol] Letting a be the first term,

$$a_5 = a \times (-3)^4 = 9$$

$$\therefore a = \frac{9}{3^4} = \frac{1}{9}$$

Therefore, the 1st term is $\frac{1}{9}$.

N 58 b

3. Obtain the common ratio of the geometric sequence whose 1st term is 243 and 8th term is $-\frac{1}{9}$.

[Sol] Letting r be the common ratio,

$$a_8 = 243r^7 = -\frac{1}{9}$$

$$3^5 \cdot r^7 = -3^{-2}$$

$$r^7 = -3^{-7}$$

$$r = -\frac{1}{3}$$

4. Obtain the 1st term of the geometric sequence whose common ratio is $-\frac{2}{3}$ and 7th term is $-\frac{64}{81}$.

[Sol] Letting a be the first term,

$$a_7 = a \times \left(-\frac{2}{3}\right)^6 = -\frac{64}{81}$$

$$a \times \left(-\frac{2}{3}\right)^6 = -\left(\frac{2^6}{3^4}\right)$$

$$\therefore a = -9$$

Therefore, the 1st term is -9 .

5. Obtain the common ratio of the geometric sequence whose 1st term is -5 and 6th term is 160.

[Sol] Letting r be the common ratio,

$$a_6 = -5r^5 = 160$$

$$r^5 = -32$$

$$r = -2$$

Geometric Sequences

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	1	-	2-

1. Obtain the 1st term and common ratio of a geometric sequence whose 5th term is 24 and 7th term is 96.

[Sol] Letting a be the first term and r be the common ratio,

$$a_5 = ar^4 = 24 \quad \dots \textcircled{1}$$

$$a_7 = ar^6 = 96 \quad \dots \textcircled{2}$$

From $\textcircled{2} \div \textcircled{1}$,

$$r^2 = 4$$

$$\therefore r = \pm 2$$

Substituting this value into $\textcircled{1}$,

$$a = \frac{3}{2}$$

2. Obtain the 1st term of the geometric sequence whose common ratio is $-\frac{1}{5}$ and 7th term is $\frac{6}{125}$.

[Sol] Letting a be the first term,

$$a_7 = a \times \left(-\frac{1}{5}\right)^6 = \frac{6}{125}$$

$$\therefore a = \frac{6}{5^3} \cdot 5^6 = 6 \cdot 5^3 = 750$$

Therefore, the 1st term is 750.

3. Obtain the common ratio of the geometric sequence whose 1st term is $2\frac{7}{9}$ and 6th term is $\frac{27}{125}$.

[Sol] Letting r be the common ratio,

$$a_6 = 2\frac{7}{9} \cdot r^5 = \frac{27}{125}$$

$$\frac{25}{9} r^5 = \frac{27}{125}$$

$$r^5 = \frac{3^3}{5^3}$$

$$r = \frac{3}{5}$$

N 59 b

4. Obtain the 1st term and common ratio of a geometric sequence whose 2nd term is $4\frac{1}{2}$ and 5th term is $\frac{1}{6}$.

[Sol] Letting a be the first term and r be the common ratio,

$$a_2 = ar = 4\frac{1}{2} = \frac{9}{2} \quad \dots \textcircled{1}$$

$$a_5 = ar^4 = \frac{1}{6} \quad \dots \textcircled{2}$$

From $\textcircled{2} \div \textcircled{1}$,

$$r^3 = \frac{1}{27}$$

$$\therefore r = \frac{1}{3}$$

Substituting this value into $\textcircled{1}$,

$$a = \frac{27}{2} \quad \left[= 13\frac{1}{2} \right]$$

5. Obtain the 1st term of the geometric sequence whose common ratio is $\frac{4}{3}$ and 4th term is $\frac{2}{3}$.

[Sol] Letting a be the first term,

$$a_4 = a \times \left(\frac{4}{3}\right)^3 = \frac{2}{3}$$

$$\therefore a = \frac{9}{32}$$

Therefore, the 1st term is $\frac{9}{32}$.

6. Obtain the 1st term and common ratio of a geometric sequence whose 2nd term is $\frac{1}{8}$ and 6th term is 32.

[Sol] Letting a be the first term and r be the common ratio,

$$a_2 = ar = \frac{1}{8} \quad \dots \textcircled{1}$$

$$a_6 = ar^5 = 32 \quad \dots \textcircled{2}$$

From $\textcircled{2} \div \textcircled{1}$,

$$r^4 = 256$$

$$r = \pm 4$$

Substituting this value into $\textcircled{1}$,

$$a = \pm \frac{1}{32}$$

Geometric Sequences

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	-	1	2-

1. Obtain the general term of a geometric sequence whose 3rd term is $\frac{1}{48}$ and 6th term is $\frac{4}{3}$. Assume each term of the sequence is a real number.

[Sol] Letting a be the first term and r be the common ratio,

$$a_3 = ar^2 = \frac{1}{48} \quad \dots \textcircled{1}$$

$$a_6 = ar^5 = \frac{4}{3} \quad \dots \textcircled{2}$$

From $\textcircled{2} \div \textcircled{1}$,

$$r^3 = 64$$

$$\therefore r = 4$$

Substituting this value into $\textcircled{1}$,

$$a = \frac{1}{768}$$

$$\therefore a_n = \frac{1}{768} \cdot 4^{n-1} = \frac{1}{3} \cdot 2^{2n-10}$$

2. Obtain the three real numbers, whose sum is 3 and product is -8 , which form a geometric sequence.

[Sol] Letting r be the common ratio; and a , ar and ar^2 be the three real numbers forming the geometric sequence,

$$a + ar + ar^2 = 3$$

$$a \cdot ar \cdot ar^2 = -8$$

$$\therefore a(1 + r + r^2) = 3 \quad \dots \textcircled{1}$$

$$ar = -2 \quad \dots \textcircled{2}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$2r^2 + 5r + 2 = 0$$

$$\therefore r = -\frac{1}{2}, -2$$

$$\therefore a = 4, 1$$

When $a = 4$ and $r = -\frac{1}{2}$, the three numbers are: 4, $-\frac{1}{2}$ and 1

When $a = 1$ and $r = -2$, the three numbers are: 1, -2 and 4

Therefore, the three real numbers are -2 , 1 and 4.

N 60 b

3. Obtain the three real numbers, whose sum is $8\frac{2}{3}$ and product is 8, which form a geometric sequence.

[Sol] Letting r be the common ratio; and a , ar and ar^2 be the three real numbers forming the geometric sequence,

$$a + ar + ar^2 = 8\frac{2}{3} \qquad a \cdot ar \cdot ar^2 = 8$$

$$\therefore a(1 + r + r^2) = \frac{26}{3} \quad \dots \textcircled{1} \qquad ar = 2 \quad \dots \textcircled{2}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$3r^2 - 10r + 3 = 0$$

$$\therefore r = \frac{1}{3}, 3$$

$$\therefore a = 6, \frac{2}{3}$$

When $a = 6$ and $r = \frac{1}{3}$, the numbers are: 6, 2, $\frac{2}{3}$

When $a = \frac{2}{3}$ and $r = 3$, the numbers are: $\frac{2}{3}$, 2, 6

Therefore, the three real numbers are $\frac{2}{3}$, 2 and 6.

4. Obtain the general term of a geometric sequence whose 2nd term is 6 and 4th term is $\frac{3}{2}$. Assume each term of the sequence is a real number.

[Sol] Letting a be the first term and r be the common ratio,

$$a_2 = ar = 6 \quad \dots \textcircled{1}$$

$$a_4 = ar^3 = \frac{3}{2} \quad \dots \textcircled{2}$$

From $\textcircled{2} \div \textcircled{1}$,

$$r^2 = \frac{1}{4}$$

$$\therefore r = \pm \frac{1}{2}$$

Substituting this value into $\textcircled{1}$,

$$a = \pm 12$$

When $r = \frac{1}{2}$ and $a = 12$, $a_n = 12 \cdot \left(\frac{1}{2}\right)^{n-1} = 3 \cdot 2^2 \cdot \left(\frac{1}{2}\right)^{n-1} = 3 \cdot \left(\frac{1}{2}\right)^{n-3}$

When $r = -\frac{1}{2}$ and $a = -12$, $a_n = -12 \cdot \left(-\frac{1}{2}\right)^{n-1} = 3 \cdot (-1)^n \cdot \left(\frac{1}{2}\right)^{n-3}$

N 61 b

$$(2) -3, 1, -\frac{1}{3}, \dots; S_6$$

$$[\text{Sol}] \quad a = -3, \quad r = -\frac{1}{3}$$

$$S_6 = \frac{-3 \left[1 - \left(-\frac{1}{3} \right)^6 \right]}{1 + \frac{1}{3}} = \frac{3}{4} \cdot (-3) \cdot \frac{728}{3^6} = -\frac{182}{81}$$

$$(3) 1, \sqrt{2}, 2, \dots; S_8$$

$$[\text{Sol}] \quad a = 1, \quad r = \sqrt{2}$$

$$S_8 = \frac{(\sqrt{2})^8 - 1}{\sqrt{2} - 1} = \frac{2^4 - 1}{\sqrt{2} - 1} = 15(\sqrt{2} + 1)$$

$$(4) -3, 2, -\frac{4}{3}, \dots; S_6$$

$$[\text{Sol}] \quad a = -3, \quad r = -\frac{2}{3}$$

$$S_6 = \frac{-3 \left[1 - \left(-\frac{2}{3} \right)^6 \right]}{1 + \frac{2}{3}} = \frac{3}{5} \cdot (-3) \cdot \frac{665}{3^6} = -\frac{133}{81}$$

$$(5) 1, \frac{1}{3}, \frac{1}{9}, \dots; S_n$$

$$[\text{Sol}] \quad a = 1, \quad r = \frac{1}{3}$$

$$S_n = \frac{1 - \left(\frac{1}{3} \right)^n}{1 - \frac{1}{3}} = \frac{3}{2} \left[1 - \left(\frac{1}{3} \right)^n \right]$$

Sums of Geometric Sequences

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	1	-	2-

Sum of a Geometric Sequence

$$\text{When } r \neq 1, \quad S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}$$

$$\text{When } r = 1, \quad S_n = na$$

(a is the first term.
 r is the common ratio.
 n is the number of terms.)

1. Using the above formulas, obtain the sum of each geometric sequence described.

- (1) A geometric sequence with first term 2, common ratio $\frac{1}{2}$, and number of terms 4.

$$[\text{Sol}] \quad S_4 = \frac{2 \left[1 - \left(\frac{1}{2} \right)^4 \right]}{1 - \frac{1}{2}} = 4 \left[1 - \left(\frac{1}{2} \right)^4 \right] = 4 \cdot \frac{15}{16} = \frac{15}{4}$$

- (2) A geometric sequence with first term 12, common ratio 1, and number of terms n .

$$S_n = 12n$$

- (3) A geometric sequence with first term 4, common ratio 4, and number of terms $n-1$.

$$[\text{Sol}] \quad S_{n-1} = \frac{4(4^{n-1}-1)}{4-1} = \frac{4}{3}(4^{n-1}-1)$$

$$\left[= \frac{4^n - 4}{3} \right]$$

N 62 b

2. In each of the following exercises, given a geometric sequence with first term a , common ratio r , n^{th} term a_n , and whose sum of the first n terms is S_n , use the given conditions to determine the value(s) indicated in parentheses.

(1) $a = 27, \quad r = \frac{1}{3}, \quad S_n = 40 \quad (n)$

$$[\text{Sol}] \quad 40 = \frac{27 \left[1 - \left(\frac{1}{3} \right)^n \right]}{1 - \frac{1}{3}}$$

$$\left(\frac{1}{3} \right)^n = \frac{1}{81}$$

$$\therefore n = 4$$

(2) $a = \frac{1}{2}, \quad a_n = 64, \quad S_n = \frac{255}{2} \quad (n, r)$

$$[\text{Sol}] \quad \begin{cases} 64 = \frac{1}{2} r^{n-1} \\ \frac{255}{2} = \frac{\frac{1}{2}(1-r^n)}{1-r} \end{cases} \quad \begin{cases} r^{n-1} = 128 & \dots \textcircled{1} \\ r^n - 255r = -254 & \dots \textcircled{2} \end{cases}$$

Substituting ① into ②,

$$128r - 255r = -254 \quad \therefore r = 2$$

Substituting this value into ①,

$$2^{n-1} = 128 \quad \therefore n = 8$$

(3) $a = 3, \quad r = 2 \quad (\text{the sum of the } 7^{\text{th}} \text{ through } 10^{\text{th}} \text{ term})$

$$[\text{Sol}] \quad S_6 = \frac{3(2^6 - 1)}{2 - 1} = 189$$

$$S_{10} = \frac{3(2^{10} - 1)}{2 - 1} = 3069$$

Therefore, the sum is:

$$3069 - 189 = 2880$$

N 63 a

Sums of Geometric Sequences

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	-	-	1-

1. Obtain the common ratio, r , and 1st term, a , of a geometric sequence with sums $S_5 = 352$ and $S_{10} = 341$. Assume that r is a real number.

[Sol] (a) When $r \neq 1$,

$$S_5 = \frac{a(r^5 - 1)}{r - 1} = 352 \quad \dots \textcircled{1}$$

$$S_{10} = \frac{a(r^{10} - 1)}{r - 1} = \boxed{341} \quad \dots \textcircled{2}$$

From $\textcircled{2} \div \textcircled{1}$,

$$r^5 + 1 = \frac{\boxed{341}}{\boxed{352}} \quad [r^{10} - 1 = (r^5 + 1)(r^5 - 1)]$$

$$\therefore r^5 = \boxed{-\frac{1}{32}} = \left(\boxed{-\frac{1}{2}}\right)^5 \quad \therefore r = \boxed{-\frac{1}{2}}$$

Substituting this into $\textcircled{1}$, $a = \boxed{512}$.

(b) When $r = 1$,

$$S_5 = \boxed{5}a = 352, \quad S_{10} = \boxed{10}a = \boxed{341}$$

However, these two equations cannot hold true simultaneously.

From (a) and (b), $a = \boxed{512}$, $r = \boxed{-\frac{1}{2}}$.

N 63 b

2. Obtain the sum S_{15} of the geometric sequence with sums $S_5 = 1$ and $S_{10} = 33$.

[Sol] Letting a be the 1st term and r be the common ratio,

(a) When $r \neq 1$,

$$S_5 = \frac{a(r^5 - 1)}{r - 1} = 1 \quad \dots \textcircled{1}$$

$$S_{10} = \frac{a(r^{10} - 1)}{r - 1} = 33 \quad \dots \textcircled{2}$$

$$\text{From } \textcircled{2} \div \textcircled{1}, \quad r^5 + 1 = 33 \quad \therefore r^5 = 32$$

In this case,

$$\begin{aligned} S_{15} &= \frac{a(r^{15} - 1)}{r - 1} = \frac{a(r^5 - 1)(r^{10} + r^5 + 1)}{r - 1} \\ &= r^{10} + r^5 + 1 = (r^5)^2 + r^5 + 1 = (32)^2 + 32 + 1 = 1057 \\ &\quad \left(\therefore \text{from } \textcircled{1}, \frac{a(r^5 - 1)}{r - 1} = 1 \right) \end{aligned}$$

(b) When $r = 1$,

$$S_5 = 5a = 1, \quad S_{10} = 10a = 33$$

However, these two equations cannot hold true simultaneously.

From (a) and (b), $S_{15} = 1057$

N 64 a

Sums of Geometric Sequences

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	-	-	1-

1. Given a geometric sequence whose sum of the first 10 terms is 4, and whose sum of the terms from the 11th through the 30th term is 48, obtain the sum of the terms from the 31st through the 60th term.

[Sol] Letting a be the first term of the geometric sequence, r be the common ratio and S_n be the sum of the first n terms,

$r = 1$ does not meet the given conditions, therefore, $r \neq 1$.

$$\begin{cases} S_{10} = \frac{a(r^{10} - 1)}{r - 1} = 4 & \dots \textcircled{1} \\ S_{30} = \frac{a(r^{30} - 1)}{r - 1} = 48 + 4 = 52 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{2} \div \textcircled{1}$, $r^{20} + r^{10} + 1 = 13$

$$(r^{10} - 3)(r^{10} + 4) = 0$$

$$\therefore r^{10} = 3$$

$$\begin{aligned} S_{60} &= \frac{a(r^{60} - 1)}{r - 1} = \frac{a(r^{30} - 1)(r^{30} + 1)}{r - 1} \\ &= 52 \times (3^3 + 1) = 1456 \end{aligned}$$

Therefore, the sum is:

$$S_{60} - S_{30} = 1456 - 52 = 1404$$

N 64 b

2. Let A be the sum of the first 10 terms of a geometric sequence, and let B be the sum of the 11th through 20th term. Obtain the sum of the 21st through 30th term. Assume that $A \neq 0$.

[Sol] Letting a be the 1st term, r be the common ratio, and C be the sum,

(a) When $r \neq 1$,

$$S_{10} = \frac{a(r^{10} - 1)}{r - 1} = A \quad \dots \textcircled{1} \quad S_{20} = \frac{a(r^{20} - 1)}{r - 1} = A + B \quad \dots \textcircled{2}$$

From $\textcircled{2} \div \textcircled{1}$,

$$r^{10} + 1 = \frac{A + B}{A} \quad \therefore r^{10} = \frac{B}{A}$$

$$\begin{aligned} S_{30} &= \frac{a(r^{30} - 1)}{r - 1} = \frac{a(r^{10} - 1)(r^{20} + r^{10} + 1)}{r - 1} \\ &= A(r^{20} + r^{10} + 1) = A \left[\left(\frac{B}{A} \right)^2 + \frac{B}{A} + 1 \right] = A + B + \frac{B^2}{A} \quad \dots \textcircled{3} \end{aligned}$$

From $\textcircled{3} - \textcircled{2}$,

$$C = S_{30} - S_{20} = \frac{B^2}{A} \quad \dots \textcircled{4}$$

(b) When $r = 1$,

$$A = 10a, \quad B = 10a, \quad C = 10a$$

Since these satisfy $\textcircled{4}$, the sum is $\frac{B^2}{A}$.

Sums of Geometric Sequences

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	—	—	—	—

1. Obtain the general term, a_n , of the sequence denoted by $S_n = 2n^2 - 3n + 1$, where S_n is the sum of the first n terms.

[Sol] In the general case,

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + \boxed{a_3}$$

$$\vdots$$

$$S_{n-1} = a_1 + a_2 + \boxed{a_3} + \cdots + a_{n-1} \boxed{1}$$

$$S_n = a_1 + a_2 + \boxed{a_3} + \cdots + a_{n-1} \boxed{1} + a_n$$

Therefore,

$$a_1 = S_{\boxed{1}}$$

$$a_n = S_n - S_{n-1} \quad (n \geq 2)$$

Thus, with this sequence,

(a) When $n \geq 2$,

$$\begin{aligned} a_n &= S_n - S_{n-1} \\ &= (2n^2 - 3n + 1) - [2(n-1)^2 - 3(n-1) + 1] \\ &= \boxed{4n - 5} \end{aligned}$$

(b) When $n = 1$,

$$a_1 = S_1 = 2 \times 1^2 - 3 \times \boxed{1} + \boxed{1} = \boxed{0}$$

Therefore,

$$a_1 = \boxed{0}$$

$$a_n = \boxed{4n - 5} \quad (n \geq 2)$$

N 65 b

2. Obtain the general term a_n of the sequence denoted by $S_n = 2n^2 - 3n$, where S_n is the sum of the first n terms.

[Sol] (a) When $n \geq 2$,

$$\begin{aligned} a_n &= S_n - S_{n-1} = 2n^2 - 3n - [2(n-1)^2 - 3(n-1)] \\ &= 4n - 5 \end{aligned}$$

(b) When $n = 1$,

$$a_1 = S_1 = -1$$

The value in (b) is the same as the value obtained when $n = 1$ is substituted into the equation obtained in (a).

Therefore, for all natural numbers n ,

$$a_n = 4n - 5$$

3. Obtain the value of c at which the sequence denoted by $S_n = n^2 + 2n + c$ becomes an arithmetic sequence. Then, obtain the 1st term and common difference which form the arithmetic sequence.

[Sol] (a) When $n \geq 2$,

$$\begin{aligned} a_n &= S_n - S_{n-1} = n^2 + 2n + c - [(n-1)^2 + 2(n-1) + c] \\ &= 2n + 1 \end{aligned}$$

(b) When $n = 1$,

$$a_1 = S_1 = 3 + c$$

In order for $\{a_n\}$ to become an arithmetic sequence, the 1st term must be $a_1 = \boxed{3}$.

Therefore, $c = \boxed{0}$.

The common difference is $\boxed{2}$.

Sums of Geometric Sequences

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	1	2	3	4

1. Obtain the general term of each sequence, where S_n is the sum of the first n terms. If the sequence is arithmetic, obtain the common difference.

(1) $S_n = n^2 - n + 1$

[Sol] (a) When $n \geq 2$,

$$\begin{aligned} a_n &= S_n - S_{n-1} \\ &= (n^2 - n + 1) - [(n-1)^2 - (n-1) + 1] \\ &= 2n - 2 \end{aligned} \quad \dots \textcircled{1}$$

(b) When $n = 1$,

$$a_1 = S_1 = 1$$

Substituting $n = 1$ into $\textcircled{1}$, $a_1 = 0$, which contradicts $a_1 = 1$.

Therefore, $a_1 = 1$, $a_n = 2n - 2$ ($n \geq 2$)

(This is not an arithmetic sequence.)

(2) $S_n = 3n^2 - 2n$

[Sol] (a) When $n \geq 2$,

$$\begin{aligned} a_n &= S_n - S_{n-1} \\ &= 3n^2 - 2n - [3(n-1)^2 - 2(n-1)] \\ &= 6n - 5 \end{aligned} \quad \dots \textcircled{1}$$

(b) When $n = 1$,

$$a_1 = S_1 = 1$$

This equals the value obtained by substituting $n = 1$ into $\textcircled{1}$.

Therefore, for all natural numbers n ,

$$a_n = 6n - 5$$

This is an arithmetic sequence whose common difference is 6.

N 66 b

2. Obtain the general term of the sequence denoted by $S_n = 2^n + 5$.

[Sol] (a) When $n \geq 2$,

$$\begin{aligned} a_n &= S_n - S_{n-1} \\ &= 2^{\boxed{n}} + 5 - (2^{\boxed{n-1}} + 5) \\ &= 2^{\boxed{n}} - 2^{\boxed{n-1}} \\ &= 2^{n-1} \quad \dots \textcircled{1} \end{aligned}$$

(b) When $n = 1$,

$$a_1 = S_1 = \boxed{7}$$

Substituting $n = 1$ into $\textcircled{1}$ gives $a_1 = \boxed{1}$, which contradicts $a_1 = \boxed{7}$.

Therefore, the general term of the sequence is:

$$a_1 = \boxed{7}, \quad a_n = \boxed{2^{n-1}} \quad (n \geq \boxed{2})$$

3. Determine the value of c at which the sequence denoted by $S_n = 3^n + c$ becomes a geometric sequence. Then, obtain the 1st term and common ratio which form the geometric sequence.

[Sol] (a) When $n \geq 2$,

$$\begin{aligned} a_n &= S_n - S_{n-1} = 3^n + c - (3^{n-1} + c) \\ &= 3^n - 3^{n-1} = 3 \cdot 3^{n-1} - 3^{n-1} \\ &= 2 \cdot 3^{n-1} \end{aligned}$$

(b) When $n = 1$,

$$a_1 = S_1 = 3 + c$$

In order for $\{a_n\}$ to become a geometric sequence, the 1st term must be $a_1 = \boxed{2}$.

Therefore, $c = \boxed{-1}$.

The common ratio is $\boxed{3}$.

Sums of Geometric Sequences

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	1	2-3	4	5-

1. Letting $\{a_k\}$ be the geometric sequence 2, 4, 8, 16, 32, 64, ..., evaluate the following sums.

$$(1) \sum_{k=1}^3 a_k = a_1 + a_2 + a_3 = 2 + 4 + 8 = 14$$

$$(2) \sum_{k=2}^4 a_k = a_2 + a_3 + a_4 = 4 + 8 + 16 = 28$$

$$(3) \sum_{k=3}^6 a_k = a_3 + a_4 + a_5 + a_6 = 8 + 16 + 32 + 64 = 120$$

$$(4) \sum_{k=4}^4 a_k = a_4 = 16$$

$$(5) \sum_{k=1}^2 a_{k+1} = a_2 + a_3 = 4 + 8 = 12$$

$$(6) \sum_{k=3}^5 a_{k-1} = a_2 + a_3 + a_4 = 4 + 8 + 16 = 28$$

$$(7) \sum_{k=1}^3 a_{2k} = a_2 + a_4 + a_6 = 4 + 16 + 64 = 84$$

$$(8) \sum_{k=1}^2 a_{3k-1} = a_2 + a_5 = 4 + 32 = 36$$

N 67 b

2. Evaluate the following sums.

$$(1) \sum_{k=1}^4 5^k = 5^1 + 5^2 + 5^3 + 5^4 = 5 + 25 + 125 + 625 = 780$$

$$(2) \sum_{k=1}^4 5^{k-1} = 5^0 + 5^1 + 5^2 + 5^3 = 1 + 5 + 25 + 125 = 156$$

$$(3) \sum_{k=1}^4 5^{k-2} = 5^{-1} + 5^0 + 5^1 + 5^2 = \frac{1}{5} + 1 + 5 + 25 = \frac{156}{5} \quad \left[= 31\frac{1}{5} \right]$$

$$(4) \sum_{k=1}^4 5^{2-k} = 5 + 5^0 + 5^{-1} + 5^{-2} = 5 + 1 + \frac{1}{5} + \frac{1}{25} = \frac{156}{25} \quad \left[= 6\frac{6}{25} \right]$$

$$(5) \sum_{k=1}^n r^k = r^1 + r^2 + \dots + r^n = \frac{r(r^n - 1)}{r - 1}$$

(Assume $1 < r$.)

$$(6) \sum_{k=1}^n ar^{k-1} = ar^0 + ar + ar^2 + \dots + ar^{n-1}$$

(Assume $1 < r$.)

$$= a(1 + r + r^2 + \dots + r^{n-1})$$

$$= \frac{a(r^n - 1)}{r - 1}$$

$$(7) \sum_{k=1}^4 (-2)^{k-1} = (-2)^0 + (-2)^1 + (-2)^2 + (-2)^3$$

$$= 1 - 2 + 4 - 8 = -5$$

$$(8) \sum_{k=1}^n (-2)^{k-1} = (-2)^0 + (-2)^1 + (-2)^2 + \dots + (-2)^{n-1}$$

$$= \frac{1 - (-2)^n}{1 - (-2)} = \frac{1 - (-2)^n}{3}$$

N 68 a

Sums of Geometric Sequences

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	1	2-3	4-5	6-

1. Evaluate the following sums.

$$(1) \sum_{k=1}^3 2^k = 2 + 2^2 + 2^3 = 2 + 4 + 8 = 14$$

$$(2) \sum_{k=1}^4 3^k = 3 + 3^2 + 3^3 + 3^4 = 3 + 9 + 27 + 81 = 120$$

$$(3) \sum_{k=1}^4 3 \cdot 2^k = 3 \cdot 2 + 3 \cdot 2^2 + 3 \cdot 2^3 + 3 \cdot 2^4 = 6 + 12 + 24 + 48 = 90$$

$$(4) \sum_{k=1}^5 2^{k+1} = 2^2 + 2^3 + 2^4 + 2^5 + 2^6 = 4 + 8 + 16 + 32 + 64 = 124$$

$$(5) \sum_{k=2}^4 2^{k+2} = 2^4 + 2^5 + 2^6 = 16 + 32 + 64 = 112$$

$$(6) \sum_{k=3}^5 4^{k-1} = 4^2 + 4^3 + 4^4 = 16 + 64 + 256 = 336$$

$$(7) \sum_{k=2}^4 3^{1-k} = 3^{-1} + 3^{-2} + 3^{-3} = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} = \frac{13}{27}$$

$$(8) \sum_{k=3}^6 2^{4-k} = 2 + 2^0 + 2^{-1} + 2^{-2} = 2 + 1 + \frac{1}{2} + \frac{1}{4} = 3\frac{3}{4} \quad \left[= \frac{15}{4} \right]$$

$$(9) \sum_{k=7}^9 (-2)^{k-3} = (-2)^4 + (-2)^5 + (-2)^6 = 16 - 32 + 64 = 48$$

$$(10) \sum_{k=2}^6 (-3)^{k-1} = (-3) + (-3)^2 + (-3)^3 + (-3)^4 + (-3)^5 \\ = -3 + 9 - 27 + 81 - 243 = -183$$

N 68 b

2. Use the sigma notation to rewrite each of the following sums.

$$(1) 3 + 9 + 27 + \dots + 3^n = \sum_{k=1}^n \boxed{3^k}$$

$$(2) 1 + 2 + 4 + 8 + \dots + 2^{n-1} = \sum_{k=1}^n \boxed{2^{k-1}}$$

$$(3) 1 + 4 + 16 + 64 + \dots + (n^{\text{th}} \text{ term}) = \sum_{k=1}^n 4^{k-1}$$

$$(4) 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + (n^{\text{th}} \text{ term}) = \sum_{k=1}^n 2^{1-k}$$

$$(5) 81 + 27 + 9 + 3 + \dots + (n^{\text{th}} \text{ term}) = \sum_{k=1}^n 3^{5-k}$$

$$(6) 50 + 10 + 2 + \frac{2}{5} + \dots + (n^{\text{th}} \text{ term}) = \sum_{k=1}^n 2 \cdot 5^{3-k}$$

$$(7) ar + ar^2 + ar^3 + \dots + ar^n = \sum_{k=1}^n ar^k$$

$$(8) a + ar + ar^2 + \dots + ar^{n-1} = \sum_{k=1}^n ar^{k-1}$$

$$(9) a + ar + ar^2 + \dots + ar^n = \sum_{k=1}^{n+1} ar^{k-1}$$

Sums of Geometric Sequences

Time : to : Date Name

100%	90%	80%	70%	60%
(mistake) 0	—	1	—	2

$$\sum_{k=1}^n (a_k + b_k) = \sum_{k=1}^n a_k + \sum_{k=1}^n b_k$$

$$\sum_{k=1}^n ca_k = c \sum_{k=1}^n a_k \quad (\text{where } c \text{ is a constant})$$

Evaluate the following sums.

Ex. $\sum_{k=1}^{10} (3k + 2^{k-1}) = 3 \sum_{k=1}^{10} k + \sum_{k=1}^{10} 2^{k-1}$

$$= 3(1 + 2 + 3 + \dots + 10) + (1 + 2 + 2^2 + \dots + 2^9)$$

$$= 3 \times \frac{10(10+1)}{2} + \frac{2^{10}-1}{2-1}$$

$$= 1188$$

(1) $\sum_{k=1}^8 (4k + 3^{k-1}) = 4 \sum_{k=1}^8 k + \sum_{k=1}^8 3^{k-1}$

$$= 4(1 + 2 + 3 + \dots + 8) + (1 + 3 + 3^2 + \dots + 3^7)$$

$$= 4 \times \frac{8(8+1)}{2} + \frac{3^8-1}{3-1}$$

$$= 3424$$

(2) $\sum_{k=1}^n (2k + 2^k) = 2 \sum_{k=1}^n k + \sum_{k=1}^n 2^k$

$$= 2 \cdot \frac{n(n+1)}{2} + \frac{2(2^n-1)}{2-1}$$

$$= n(n+1) + 2(2^n-1)$$

$$= 2^{n+1} + n(n+1) - 2$$

N 69 b

$$\begin{aligned}
 (3) \quad \sum_{k=1}^n (5k + 3^{k+1}) &= 5 \sum_{k=1}^n k + 3 \sum_{k=1}^n 3^k \\
 &= 5 \cdot \frac{n(n+1)}{2} + 3 \cdot \frac{3^n - 1}{3 - 1} \\
 &= \frac{5}{2}n(n+1) + \frac{9}{2}(3^n - 1) \\
 &= \frac{5n^2 + 5n + 3^{n+2} - 9}{2}
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad \sum_{k=1}^n (nk + 3^k) &= n \sum_{k=1}^n k + \sum_{k=1}^n 3^k \\
 &= n \cdot \frac{n(n+1)}{2} + \frac{3(3^n - 1)}{3 - 1} \\
 &= \frac{n^3 + n^2 + 3^{n+1} - 3}{2}
 \end{aligned}$$

$$\begin{aligned}
 (5) \quad \sum_{k=1}^n \frac{2^k + 3^k}{5^k} &= \sum_{k=1}^n \left(\frac{2}{5}\right)^k + \sum_{k=1}^n \left(\frac{3}{5}\right)^k \\
 &= \frac{\frac{2}{5} \left[1 - \left(\frac{2}{5}\right)^n\right]}{1 - \frac{2}{5}} + \frac{\frac{3}{5} \left[1 - \left(\frac{3}{5}\right)^n\right]}{1 - \frac{3}{5}} \\
 &= \frac{2}{3} \left[1 - \left(\frac{2}{5}\right)^n\right] + \frac{3}{2} \left[1 - \left(\frac{3}{5}\right)^n\right] \\
 &= \frac{13}{6} - \frac{2}{3} \left(\frac{2}{5}\right)^n - \frac{3}{2} \left(\frac{3}{5}\right)^n
 \end{aligned}$$

Sums of Geometric Sequences

Time : to : Date Name

100%	90%	80%	70%	60%~
(mistakes) 0	-	-	1	2-

1. Given a geometric sequence with 2nd term 16 and 6th term 81, and whose common ratio is a real number, complete the following exercises.

- (1) Obtain the 1st term and the common ratio.

[Sol] Letting a be the first term and r be the common ratio,

$$ar = 16 \quad \dots \textcircled{1}$$

$$ar^5 = 81 \quad \dots \textcircled{2}$$

$$\text{From } \textcircled{2} \div \textcircled{1}, r^4 = \frac{81}{16} \quad \therefore r = \pm \frac{3}{2}$$

$$\text{Substituting these into } \textcircled{1}, \begin{cases} r = \pm \frac{3}{2} \\ a = \pm \frac{32}{3} \end{cases}$$

- (2) Obtain the sum, S_n , through the n^{th} term.

[Sol] When $a = \frac{32}{3}, r = \frac{3}{2},$

$$S_n = \frac{\frac{32}{3} \left[\left(\frac{3}{2} \right)^n - 1 \right]}{\frac{3}{2} - 1} = \frac{64}{3} \left[\left(\frac{3}{2} \right)^n - 1 \right]$$

When $a = -\frac{32}{3}, r = -\frac{3}{2},$

$$S_n = \frac{-\frac{32}{3} \left[1 - \left(-\frac{3}{2} \right)^n \right]}{1 + \frac{3}{2}} = -\frac{64}{15} \left[1 - \left(-\frac{3}{2} \right)^n \right]$$

N 70 b

2. Obtain the general term, a_n , when $S_n = n^3$.

$$\begin{aligned} \text{[Sol]} \text{ When } n \geq 2, \quad a_n &= S_n - S_{n-1} \\ &= n^3 - (n-1)^3 \\ &= 3n^2 - 3n + 1 \quad \dots \textcircled{1} \end{aligned}$$

$$\text{When } n = 1, \quad a_1 = S_1 = 1.$$

Therefore, $\textcircled{1}$ holds true when $n \geq 1$.

$$\therefore a_n = 3n^2 - 3n + 1 \quad (n \geq 1)$$

3. Given a geometric sequence whose sum of the first 10 terms is 2, and whose sum from the 11th through 30th term is 12, obtain the sum from the 31st through 60th term.

[Sol] Letting a be the first term and r be the common ratio,

(a) When $r \neq 1$,

$$S_{10} = \frac{a(r^{10} - 1)}{r - 1} = 2 \quad \dots \textcircled{1} \quad S_{30} = \frac{a(r^{30} - 1)}{r - 1} = 2 + 12 = 14 \quad \dots \textcircled{2}$$

$$\text{From } \textcircled{2} \div \textcircled{1}, \quad r^{20} + r^{10} + 1 = 7$$

$$(r^{10} + 3)(r^{10} - 2) = 0$$

$$\therefore r^{10} = 2$$

$$S_{60} = \frac{a(r^{60} - 1)}{r - 1} = \frac{a(r^{30} - 1)(r^{30} + 1)}{r - 1}$$

$$= 14(2^3 + 1) = 126$$

$$S = S_{60} - S_{30} = 126 - 14 = 112$$

(b) When $r = 1$, $S_{10} = 10a = 2$, $S_{30} = 30a = 14$

However, these two equations do not hold true simultaneously.

$$\therefore S = 112$$

Various Sequences

Time : to : Date Name

100%	90%	80%	70%	60%~
(mistakes) 0	-	-	1	2-

1. Evaluate the following sum.

$$1^2 + 2^2 + 3^2 + \dots + n^2$$

[Sol]

Using the identity $(k+1)^3 - k^3 = 3k^2 + 3k + 1$,

$$\text{When } k = 1, \quad 2^3 - 1^3 = 3 \cdot 1^2 + 3 \cdot 1 + 1$$

$$\text{When } k = 2, \quad 3^3 - 2^3 = 3 \cdot 2^2 + 3 \cdot 2 + 1$$

$$\text{When } k = 3, \quad \boxed{4^3} - 3^3 = 3 \cdot \boxed{3^2} + 3 \cdot \boxed{3} + \boxed{1}$$

⋮

⋮

$$\text{When } k = n, \quad (n + \boxed{1})^3 - \boxed{n}^3 = 3 \cdot \boxed{n^2} + 3 \cdot \boxed{n} + 1$$

Adding the corresponding sides of these equalities,

$$\begin{aligned} \text{LHS} &= (2^3 - 1^3) + (3^3 - 2^3) + (\boxed{4^3} - 3^3) + \dots + [(n + \boxed{1})^3 - \boxed{n}^3] \\ &= (n + \boxed{1})^3 - \boxed{1}^3 \end{aligned}$$

$$\text{RHS} = 3(1^2 + 2^2 + \dots + \boxed{n}^2) + 3(1 + 2 + \dots + \boxed{n}) + \boxed{n}$$

$$\therefore 3(1^2 + 2^2 + \dots + \boxed{n}^2) = (n + \boxed{1})^3 - \boxed{1}^3 - 3(1 + 2 + \dots + \boxed{n}) - \boxed{n}$$

$$= (n + \boxed{1})^3 - \frac{3n(\boxed{n+1})}{2} - (n+1)$$

$$= \frac{2(n+1)^3 - 3n(\boxed{n+1}) - 2(n+1)}{2}$$

$$= \frac{(n+1)(2n^2 + \boxed{n})}{2}$$

$$= \frac{n(n+1)(\boxed{2n+1})}{2}$$

$$\therefore 1^2 + 2^2 + \dots + n^2 = \boxed{\frac{n(n+1)(2n+1)}{6}}$$

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

2. Evaluate the following sums.

(1) $1^2 + 2^2 + 3^2 + \dots + 20^2$

[Sol] $\sum_{k=1}^{20} k^2 = \frac{20(20+1)(2 \cdot 20 + 1)}{6} = 2870$

(2) $11^2 + 12^2 + 13^2 + \dots + 20^2$

[Sol] $\sum_{k=1}^{20} k^2 - \sum_{k=1}^{10} k^2 = 2870 - \frac{10(10+1)(2 \cdot 10 + 1)}{6}$
 $= 2870 - 385 = 2485$

3. Evaluate the sum $\sum_{k=1}^n (k^2 + 3k + 2)$.

[Sol] $\sum_{k=1}^n (k^2 + 3k + 2) = \sum_{k=1}^n k^2 + 3 \sum_{k=1}^n k + \sum_{k=1}^n 2$
 $= \frac{n(n+1)(2n+1)}{6} + 3 \cdot \frac{n(n+1)}{2} + 2n$
 $= \frac{n}{6} (2n^2 + 3n + 1 + 9n + 9 + 12)$
 $= \frac{n}{3} (n^2 + 6n + 11)$

Various Sequences

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	-	1	2

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

1. Evaluate the sum $\sum_{k=1}^n (k^2 - 3k + 2)$.

$$\begin{aligned}
 [\text{Sol}] \quad & \sum_{k=1}^n k^2 - 3 \sum_{k=1}^n k + 2n \\
 &= \frac{n(n+1)(2n+1)}{6} - \frac{3n(n+1)}{2} + 2n \\
 &= \frac{n}{3}(n^2 - 3n + 2) \\
 &= \frac{n}{3}(n-1)(n-2)
 \end{aligned}$$

2. Evaluate the sum $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n+2)$.

[Sol] The k^{th} term is $k(k+2)$.

$$\begin{aligned}
 \therefore S_n &= \sum_{k=1}^n k(k+2) \\
 &= \sum_{k=1}^n k^2 + 2 \sum_{k=1}^n k \\
 &= \frac{n(n+1)(2n+1)}{6} + 2 \cdot \frac{n(n+1)}{2} \\
 &= \frac{n}{6}(n+1)(2n+7)
 \end{aligned}$$

N 72 b

3. Evaluate the sum $1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \dots + (n^{\text{th}} \text{ term})$.

[Sol] The k^{th} term is $(2k-1)(\boxed{2k+1}) = \boxed{4k^2-1}$.

$$\begin{aligned}\therefore S_n &= \sum_{k=1}^n (4k^2 - 1) \\ &= 4 \sum_{k=1}^n k^2 - n \\ &= 4 \cdot \frac{n(n+1)(2n+1)}{6} - n \\ &= \frac{n}{3}(4n^2 + 6n - 1)\end{aligned}$$

4. Evaluate the sum $1^2 + 3^2 + 5^2 + \dots + (n^{\text{th}} \text{ term})$.

[Sol] The k^{th} term is $(2k-1)^2$.

$$\begin{aligned}\therefore S_n &= \sum_{k=1}^n (2k-1)^2 \\ &= \sum_{k=1}^n (4k^2 - 4k + 1) \\ &= 4 \sum_{k=1}^n k^2 - 4 \sum_{k=1}^n k + n \\ &= 4 \cdot \frac{n(n+1)(2n+1)}{6} - 4 \cdot \frac{n(n+1)}{2} + n \\ &= \frac{n}{3}(4n^2 - 1) \\ &= \frac{n}{3}(2n-1)(2n+1)\end{aligned}$$

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	-	1	2

1. Evaluate the following sums.

$$(1) 1 \cdot 4 + 4 \cdot 7 + 7 \cdot 10 + \cdots + (n^{\text{th}} \text{ term})$$

$$[\text{Sol}] a_k = (3k - 2)(3k + 1)$$

$$= 9k^2 - 3k - 2$$

$$\therefore \sum_{k=1}^n a_k = 9 \sum_{k=1}^n k^2 - 3 \sum_{k=1}^n k - 2n$$

$$= 9 \cdot \frac{n(n+1)(2n+1)}{6} - 3 \cdot \frac{n(n+1)}{2} - 2n$$

$$= n(3n^2 + 3n - 2)$$

$$(2) 1^2 + 4^2 + 7^2 + \cdots + (n^{\text{th}} \text{ term})$$

$$[\text{Sol}] a_k = (3k - 2)^2$$

$$= 9k^2 - 12k + 4$$

$$\therefore \sum_{k=1}^n a_k = 9 \sum_{k=1}^n k^2 - 12 \sum_{k=1}^n k + 4n$$

$$= 9 \cdot \frac{n(n+1)(2n+1)}{6} - 12 \cdot \frac{n(n+1)}{2} + 4n$$

$$= \frac{n}{2}(6n^2 - 3n - 1)$$

N 73 b

2. Evaluate the sum of the following sequence.

$$1, \quad 1+2, \quad 1+2+3, \quad 1+2+3+4, \quad \dots, \quad 1+2+\dots+n$$

[Sol] The k^{th} term is $1+2+3+\dots+k = \boxed{\frac{k(k+1)}{2}}$.

$$\begin{aligned}\therefore \sum_{k=1}^n a_k &= \frac{1}{2} \sum_{k=1}^n k^2 + \frac{1}{2} \sum_{k=1}^n k \\ &= \frac{1}{2} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{1}{2} \cdot \frac{n(n+1)}{2} \\ &= \frac{n}{6}(n+1)(n+2)\end{aligned}$$

3. Evaluate the following sum.

$$1 \cdot n + 2 \cdot (n-1) + 3 \cdot (n-2) + \dots + (n-1) \cdot 2 + n \cdot 1$$

[Sol] The k^{th} term is $k[n - (k-1)]$.

$$\begin{aligned}a_k &= nk - k^2 + k = (n+1)k - k^2 \\ \therefore \sum_{k=1}^n a_k &= (n+1) \sum_{k=1}^n k - \sum_{k=1}^n k^2 \\ &= \frac{n}{2}(n+1)^2 - \frac{n(n+1)(2n+1)}{6} \\ &= \frac{n}{6}(n+1)(n+2)\end{aligned}$$

Various Sequences

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	—	—	—	—

1. Evaluate the sum $\sum_{k=1}^n k^3 = 1^3 + 2^3 + 3^3 + \dots + n^3$.

[Sol] Using the identity $(k+1)^4 - k^4 = 4k^3 + 6k^2 + 4k + 1$,

$$\text{When } k = 1, \quad 2^4 - 1^4 = 4 \cdot 1^3 + 6 \cdot 1^2 + 4 \cdot 1 + 1$$

$$\text{When } k = 2, \quad 3^4 - 2^4 = 4 \cdot 2^3 + 6 \cdot 2^2 + 4 \cdot 2 + 1$$

$$\text{When } k = 3, \quad \boxed{4^4} - \boxed{3^4} = \boxed{4 \cdot 3^3 + 6 \cdot 3^2 + 4 \cdot 3 + 1}$$

$$\vdots$$

$$\vdots$$

$$\text{When } k = n, \quad (n+1)^4 - n^4 = \boxed{4n^3 + 6n^2 + 4n + 1}$$

Adding the corresponding sides of these equalities,

$$\begin{aligned} \text{LHS} &= (2^4 - 1^4) + (3^4 - 2^4) + (4^4 - 3^4) + \dots + [(n+1)^4 - n^4] \\ &= (n+1)^4 - 1^4 \end{aligned}$$

$$\begin{aligned} \text{RHS} &= 4 \sum_{k=1}^n k^3 + 6 \sum_{k=1}^n k^2 + 4 \sum_{k=1}^n k + n \\ &= 4 \sum_{k=1}^n k^3 + 6 \cdot \frac{n(n+1)(\boxed{2n+1})}{6} + 4 \cdot \frac{n(\boxed{n+1})}{2} + n \end{aligned}$$

$$\begin{aligned} \therefore 4 \sum_{k=1}^n k^3 &= (n+1)^4 - 1 - n(n+1)(\boxed{2n+1}) - 2n(\boxed{n+1}) - n \\ &= (n+1)[(n+1)^3 - n(\boxed{2n+1}) - 2n - 1] \\ &= (n+1)(n^3 + \boxed{n^2}) \\ &= n^2(\boxed{n+1})^2 \end{aligned}$$

$$\therefore \sum_{k=1}^n k^3 = \boxed{\frac{n^2(n+1)^2}{4}}$$

N 74 b

$$\sum_{k=1}^n k^3 = \left[\frac{n(n+1)}{2} \right]^2$$

2. Evaluate the following sums.

$$\begin{aligned} (1) \quad \sum_{k=1}^n (4k^3 - 6k^2 + 3) &= 4 \sum_{k=1}^n k^3 - 6 \sum_{k=1}^n k^2 + 3n \\ &= 4 \cdot \frac{n^2(n+1)^2}{4} - 6 \cdot \frac{n(n+1)(2n+1)}{6} + 3n \\ &= n(n^3 - 2n + 2) \end{aligned}$$

$$(2) \quad 1^3 + 3^3 + 5^3 + \dots + (n^{\text{th}} \text{ term})$$

[Sol] The k^{th} term is $(2k-1)^3 = 8k^3 - 12k^2 + 6k - 1$.

$$\begin{aligned} \therefore \sum_{k=1}^n (2k-1)^3 &= 8 \sum_{k=1}^n k^3 - 12 \sum_{k=1}^n k^2 + 6 \sum_{k=1}^n k - n \\ &= 8 \cdot \frac{n^2(n+1)^2}{4} - 12 \cdot \frac{n(n+1)(2n+1)}{6} + 6 \cdot \frac{n(n+1)}{2} - n \\ &= n^2(2n^2 - 1) \end{aligned}$$

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	-	-	1-

1. Evaluate the following sum.

$$S_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)}$$

[Sol] Using the identity $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$,

$$\text{When } k = 1, \quad \frac{1}{1 \cdot 2} = \frac{1}{1} - \frac{1}{2}$$

$$\text{When } k = 2, \quad \frac{1}{2 \cdot 3} = \frac{1}{2} - \frac{1}{3}$$

$$\text{When } k = 3, \quad \frac{1}{3 \cdot 4} = \frac{1}{3} - \frac{1}{4}$$

$$\text{When } k = 4, \quad \frac{1}{4 \cdot 5} = \frac{1}{4} - \frac{1}{5}$$

$$\vdots \quad \quad \quad \vdots$$

$$\text{When } k = n-1, \quad \frac{1}{(n-1)n} = \frac{1}{n-1} - \frac{1}{n}$$

$$\text{When } k = n, \quad \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

Adding the corresponding sides,

$$\text{RHS} = \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

$$S_n = \boxed{1 - \frac{1}{n+1}}$$

$$= \frac{\boxed{n}}{n+1}$$

N 75 b

2. Evaluate the following sum.

$$S_n = \frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{n(n+2)}$$

[Sol] Using the identity $\frac{1}{k(k+2)} = \frac{1}{2} \left(\frac{1}{k} - \frac{1}{k+2} \right)$,

$$\text{When } k = 1, \quad \frac{1}{1 \cdot 3} = \frac{1}{2} \left(\frac{1}{1} - \frac{1}{3} \right)$$

$$\text{When } k = 2, \quad \frac{1}{2 \cdot 4} = \frac{1}{2} \left(\frac{1}{2} - \frac{1}{4} \right)$$

$$\text{When } k = 3, \quad \frac{1}{3 \cdot 5} = \frac{1}{2} \left(\frac{1}{3} - \frac{1}{5} \right)$$

$$\text{When } k = 4, \quad \frac{1}{4 \cdot 6} = \frac{1}{2} \left(\frac{1}{4} - \frac{1}{6} \right)$$

$$\vdots \quad \quad \quad \vdots$$

$$\text{When } k = n-1, \quad \frac{1}{(n-1)(n+1)} = \frac{1}{2} \left(\frac{1}{n-1} - \frac{1}{n+1} \right)$$

$$\text{When } k = n, \quad \frac{1}{n(n+2)} = \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+2} \right)$$

Adding the corresponding sides,

$$S_n = \frac{1}{2} \left(1 + \frac{1}{2} - \boxed{\frac{1}{n+1}} - \boxed{\frac{1}{n+2}} \right)$$

$$= \frac{n(\boxed{3n+5})}{4(n+1)(n+2)}$$

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	-	-	1-

1. Evaluate the following sum.

$$S_n = \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots + \frac{1}{(2n-1)(2n+1)}$$

$$[\text{Sol}] a_k = \frac{1}{(2k-1)(2k+1)} = \frac{1}{2} \left(\frac{1}{2k-1} - \frac{1}{2k+1} \right)$$

$$\text{When } k = 1, \quad \frac{1}{1 \cdot 3} = \frac{1}{2} \left(\frac{1}{1} - \frac{1}{3} \right)$$

$$\text{When } k = 2, \quad \frac{1}{3 \cdot 5} = \frac{1}{2} \left(\frac{1}{3} - \frac{1}{5} \right)$$

⋮

⋮

$$\text{When } k = n, \quad \frac{1}{(2n-1)(2n+1)} = \frac{1}{2} \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right)$$

Adding the corresponding sides,

$$\begin{aligned}
 S_n &= \frac{1}{2} \left(1 - \frac{1}{2n+1} \right) \\
 &= \frac{n}{2n+1}
 \end{aligned}$$

N 76 b

2. Evaluate the following sum.

$$S_n = \frac{1}{1 \cdot 4} + \frac{1}{2 \cdot 5} + \frac{1}{3 \cdot 6} + \cdots + \frac{1}{n(n+3)}$$

$$[\text{Sol}] \quad a_k = \frac{1}{k(k+3)} = \frac{1}{3} \left(\frac{1}{k} - \frac{1}{k+3} \right)$$

$$\text{When } k = 1, \quad \frac{1}{1 \cdot 4} = \frac{1}{3} \left(\frac{1}{1} - \frac{1}{4} \right)$$

$$\text{When } k = 2, \quad \frac{1}{2 \cdot 5} = \frac{1}{3} \left(\frac{1}{2} - \frac{1}{5} \right)$$

$$\text{When } k = 3, \quad \frac{1}{3 \cdot 6} = \frac{1}{3} \left(\frac{1}{3} - \frac{1}{6} \right)$$

$$\text{When } k = 4, \quad \frac{1}{4 \cdot 7} = \frac{1}{3} \left(\frac{1}{4} - \frac{1}{7} \right)$$

$$\vdots$$

$$\vdots$$

$$\text{When } k = n-3, \quad \frac{1}{(n-3)n} = \frac{1}{3} \left(\frac{1}{n-3} - \frac{1}{n} \right)$$

$$\text{When } k = n-2, \quad \frac{1}{(n-2)(n+1)} = \frac{1}{3} \left(\frac{1}{n-2} - \frac{1}{n+1} \right)$$

$$\text{When } k = n-1, \quad \frac{1}{(n-1)(n+2)} = \frac{1}{3} \left(\frac{1}{n-1} - \frac{1}{n+2} \right)$$

$$\text{When } k = n, \quad \frac{1}{n(n+3)} = \frac{1}{3} \left(\frac{1}{n} - \frac{1}{n+3} \right)$$

Adding the corresponding sides,

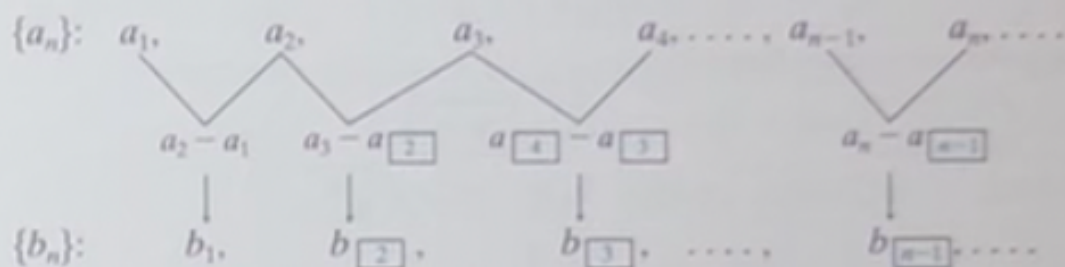
$$\begin{aligned} S_n &= \frac{1}{3} \left(1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3} \right) \\ &= \frac{n(11n^2 + 48n + 49)}{18(n+1)(n+2)(n+3)} \end{aligned}$$

Various Sequences

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	-	1	-	2-

Given a sequence $\{a_n\}$: $a_1, a_2, a_3, a_4, \dots, a_{n-1}, a_n, \dots$, letting $\{b_n\}$ be a sequence whose terms are derived by taking the difference of the terms in $\{a_n\}$,



$\{b_n\}$ is called the *sequence of first-order differences* of the original sequence.

$$b_n = a_{n+1} - a_n \quad (n = 1, 2, 3, \dots)$$

Answer: 2, 4, 3, n-1, 2, 3, n-1

- For each given sequence $\{a_n\}$, derive the first 5 terms of the sequence of first-order differences, $\{b_n\}$, and then obtain b_n .

Ex. $\{a_n\}$: 4, 8, 13, 19, 26, 34, ...

$\{b_n\}$: 4, 5, 6, 7, 8, ... $\therefore b_n = n + 3$

(1) $\{a_n\}$: 5, 8, 12, 17, 23, 30, ...

$\{b_n\}$: 3, 4, 5, 6, 7, ... $\therefore b_n = n + 2$

(2) $\{a_n\}$: 1, 2, 4, 7, 11, 16, ...

$\{b_n\}$: 1, 2, 3, 4, 5, ... $\therefore b_n = n$

(3) $\{a_n\}$: 3, 6, 11, 18, 27, 38, ...

$\{b_n\}$: 3, 5, 7, 9, 11, ... $\therefore b_n = 2n + 1$

(4) $\{a_n\}$: 1, 6, 16, 31, 51, 76, ...

$\{b_n\}$: 5, 10, 15, 20, 25, ... $\therefore b_n = 5n$

Given a sequence $\{a_n\}$, letting $\{b_n\}$ be its sequence of first-order differences,

$$\begin{aligned} \text{When } n \geq 2, \quad & a_2 - a_1 = b_1 \\ & a_3 - a_2 = b_2 \\ & a_4 - \boxed{a_3} = \boxed{b_3} \\ & \vdots \\ & a_n - a_{n-1} = \boxed{b_{n-1}} \end{aligned}$$

Adding the corresponding sides,

$$\boxed{a_n - a_1} = b_1 + b_2 + \cdots + \boxed{b_{n-1}}$$

$$\begin{aligned} \text{Therefore, } a_n &= \boxed{a_1} + (b_1 + b_2 + \cdots + \boxed{b_{n-1}}) \\ &= \boxed{a_1} + \sum_{k=1}^{n-1} b_k \quad (n \geq 2) \end{aligned}$$

Thus, the n^{th} term (where $n \geq 2$) of the sequence $\{a_n\}$ is denoted by using its $\boxed{1^{\text{st}}}$ term, and the sum of the first $(n-1)$ terms of its sequence of differences.

ANSWERS: $a_1, b_1, b_2, b_3, \dots, b_{n-1}, a_2 - a_1, a_3 - a_2, a_4 - a_3, \dots, a_n - a_{n-1}$

2. Given sequence $\{a_n\}$, deriving its first-order sequence of differences, obtain the general term, a_n .

$$[\text{Sol}] \{a_n\}: \quad 1, 3, 7, 13, 21, 31, \dots \quad \dots \textcircled{1}$$

$$\{b_n\}: \quad 2, 4, 6, 8, 10, \dots$$

$$\therefore b_n = 2n$$

$$\begin{aligned} \text{When } n \geq 2, \quad a_n &= a_1 + \sum_{k=1}^{n-1} b_k = 1 + \sum_{k=1}^{n-1} 2k \\ &= 1 + 2 \cdot \frac{n(n-1)}{2} = \boxed{n^2 - n + 1} \quad \dots \textcircled{2} \end{aligned}$$

$$\text{When } n = 1, \text{ from } \textcircled{1}, \quad a_1 = 1.$$

Substituting $n = 1$ into $\textcircled{2}$, $a_1 = 1$. These values are consistent.

Therefore, for all natural numbers n ,

$$a_n = \boxed{n^2 - n + 1}$$

Various Sequences

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	—	—	1	2

1. Given the sequence $\{a_n\}$: 2, 5, 10, 17, 26, 37, 50, ..., complete the following exercises.

- (1) Obtain the general term, a_n .

[Sol] From $\{a_n\}$: 2, 5, 10, 17, 26, 37, 50, ①

Deriving the sequence of first-order differences,

$$\{b_n\}: 3, 5, 7, 9, 11, 13, \dots$$

$$\therefore b_n = 3 + 2(n - 1) = 2n + 1$$

$$\begin{aligned}\text{When } n \geq 2, \quad a_n &= a_1 + \sum_{k=1}^{n-1} b_k = 2 + \sum_{k=1}^{n-1} (2k + 1) \\ &= 2 + 2 \cdot \frac{n(n-1)}{2} + (n-1) \\ &= n^2 + 1 \quad \dots \text{②}\end{aligned}$$

When $n = 1$, from ①, $a_1 = 2$.

Substituting $n = 1$ into ②, $a_1 = 2$. These values are consistent.

Therefore, for all natural numbers n ,

$$a_n = n^2 + 1$$

- (2) Obtain the sum of the first n terms, S_n .

$$\begin{aligned}\text{[Sol]} \quad S_n &= \sum_{k=1}^n a_k \\ &= \sum_{k=1}^n (k^2 + 1) \\ &= \frac{n}{6}(n+1)(2n+1) + n \\ &= \frac{n}{6}(2n^2 + 3n + 7)\end{aligned}$$

N 78 b

2. Given the sequence $\{a_n\}$: 3, 6, 11, 18, 27, ..., complete the following exercises.

- (1) Obtain the general term, a_n .

[Sol] From $\{a_n\}$: 3, 6, 11, 18, 27, ...

Deriving the sequence of first-order differences, ... ①

$$\{b_n\}: 3, 5, 7, 9, \dots$$

$$\therefore b_n = 3 + 2(n-1) = 2n + 1$$

$$\text{When } n \geq 2, \quad a_n = a_1 + \sum_{k=1}^{n-1} b_k = 3 + \sum_{k=1}^{n-1} (2k + 1)$$

$$= 3 + 2 \cdot \frac{n(n-1)}{2} + (n-1)$$

$$= n^2 + 2 \quad \dots \text{②}$$

When $n = 1$, from ①, $a_1 = 3$.

Substituting $n = 1$ into ②, $a_1 = 3$. These values are consistent.

Therefore, for all natural numbers n ,

$$a_n = n^2 + 2$$

Obtain the sum of the first n terms, S_n .

$$[\text{Sol}] \quad S_n = \sum_{k=1}^n a_k$$

$$= \sum_{k=1}^n (k^2 + 2)$$

$$= \frac{n(n+1)(2n+1)}{6} + 2n$$

$$= \frac{n}{6} (2n^2 + 3n + 13)$$

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	-	1	2

1. Given the sequence $\{a_n\}$: 1, 3, 7, 15, 31, 63, ..., complete the following exercises.

- (1) Obtain the general term, a_n .

[Sol] From $\{a_n\}$: 1, 3, 7, 15, 31, 63, ①

Deriving the sequence of first-order differences,

$$\{b_n\}: 2, 4, 8, 16, \dots$$

$$\therefore b_n = 2^n$$

$$\begin{aligned} \text{When } n \geq 2, \quad a_n &= a_1 + \sum_{k=1}^{n-1} b_k = 1 + \sum_{k=1}^{n-1} 2^k \\ &= 1 + \frac{2(2^{n-1} - 1)}{2 - 1} \\ &= 2^n - 1 \quad \dots \text{②} \end{aligned}$$

When $n = 1$, from ①, $a_1 = 1$.

Substituting $n = 1$ into ②, $a_1 = 1$. These values are consistent.

Therefore, for all natural numbers n ,

$$a_n = 2^n - 1$$

- (2) Obtain the sum of the first n terms, S_n .

$$\begin{aligned} \text{[Sol]} \quad S_n &= \sum_{k=1}^n a_k \\ &= \sum_{k=1}^n (2^k - 1) \\ &= \frac{2(2^n - 1)}{2 - 1} - n \\ &= 2^{n+1} - n - 2 \end{aligned}$$

N 79 b

2. Given the sequence $\{a_n\}$: 2, 3, 0, 9, -18, 63, ..., complete the following exercises.

- (1) Obtain the general term, a_n .

[Sol] From $\{a_n\}$: 2, 3, 0, 9, -18, 63, ①

Deriving the sequence of first-order differences,

$$\{b_n\}: 1, -3, \boxed{9}, \boxed{-27}, \dots$$

$$\therefore b_n = (-3)^{n-1}$$

$$\begin{aligned} \text{When } n \geq 2, \quad a_n &= a_1 + \sum_{k=1}^{n-1} b_k = 2 + \sum_{k=1}^{n-1} (-3)^{k-1} \\ &= 2 + \frac{1 - (-3)^{n-1}}{1 + 3} \\ &= \frac{9}{4} - \frac{(-3)^{n-1}}{4} \\ &= \frac{9}{4} + \frac{(-3)^n}{12} \quad \dots \text{②} \end{aligned}$$

When $n = 1$, from ①, $a_1 = 2$.

Substituting $n = 1$ into ②, $a_1 = 2$. These values are consistent.

Therefore, for all natural numbers n ,

$$a_n = \frac{9}{4} + \frac{(-3)^n}{12}$$

- (2) Obtain the sum of the first n terms, S_n .

$$\begin{aligned} \text{[Sol]} \quad S_n &= \sum_{k=1}^n a_k \\ &= \sum_{k=1}^n \left[\frac{9}{4} + \frac{(-3)^k}{12} \right] \\ &= \frac{9}{4}n + \frac{1}{12} \cdot \frac{-3[1 - (-3)^n]}{1 + 3} \\ &= \frac{9}{4}n + \frac{(-3)^n}{16} - \frac{1}{16} \end{aligned}$$

Various Sequences

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	-	-	-	1-

1. Obtain the sum of the first n terms, S_n , of the following sequence.

$$\frac{1}{2 \cdot 6}, \frac{1}{4 \cdot 8}, \frac{1}{6 \cdot 10}, \frac{1}{8 \cdot 12}, \dots$$

$$[\text{Sol}] a_k = \frac{1}{2k(2k+4)} = \frac{1}{8} \left(\frac{1}{k} - \frac{1}{k+2} \right)$$

$$\text{When } k = 1, \quad \frac{1}{2 \cdot 6} = \frac{1}{8} \left(\frac{1}{1} - \frac{1}{3} \right)$$

$$\text{When } k = 2, \quad \frac{1}{4 \cdot 8} = \frac{1}{8} \left(\frac{1}{2} - \frac{1}{4} \right)$$

$$\text{When } k = 3, \quad \frac{1}{6 \cdot 10} = \frac{1}{8} \left(\frac{1}{3} - \frac{1}{5} \right)$$

⋮

⋮

$$\text{When } k = n, \quad \frac{1}{2n(2n+4)} = \frac{1}{8} \left(\frac{1}{n} - \frac{1}{n+2} \right)$$

Adding the corresponding sides,

$$\begin{aligned} S_n = \sum_{k=1}^n a_k &= \frac{1}{8} \left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right) \\ &= \frac{n(3n+5)}{16(n+1)(n+2)} \end{aligned}$$

N 80 b

2. Given sequence $\{a_n\}$, obtain the general term, a_n , and the sum of the first n terms, S_n .

$$\{a_n\}: \quad 3, 7, 13, 21, 31, \dots \quad \dots \textcircled{1}$$

[Sol] Deriving the sequence of first-order differences,

$$\{b_n\}: \quad 4, 6, 8, 10, \dots$$

$$\therefore b_n = 2n + 2$$

$$\begin{aligned} \text{When } n \geq 2, \quad a_n &= a_1 + \sum_{k=1}^{n-1} b_k = 3 + \sum_{k=1}^{n-1} (2k + 2) \\ &= 3 + 2 \cdot \frac{n(n-1)}{2} + 2(n-1) \\ &= n^2 + n + 1 \quad \dots \textcircled{2} \end{aligned}$$

When $n = 1$, from $\textcircled{1}$, $a_1 = 3$.

Substituting $n = 1$ into $\textcircled{2}$, $a_1 = 3$. These values are consistent.

Therefore, for all natural numbers n ,

$$a_n = n^2 + n + 1$$

$$\begin{aligned} S_n &= \sum_{k=1}^n a_k \\ &= \sum_{k=1}^n (k^2 + k + 1) \\ &= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} + n \\ &= \frac{n}{3}(n^2 + 3n + 5) \end{aligned}$$

Infinite Sequences

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	1	2	3	4

Given that the general term of a sequence is $n + 2$, let's analyze how the terms of the sequence progress as n increases.

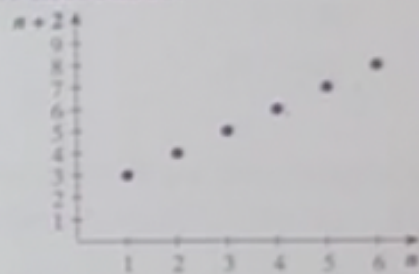
Letting $a_n = n + 2$,

$$a_1 = 3$$

$$a_2 = 4$$

$$a_3 = 5$$

⋮



Note: As n increases, each term in the sequence increases. Therefore, as n approaches infinity, the sequence **diverges** to *positive infinity*.

- Given that the general term of a sequence is $3n$, analyze how the terms of the sequence progress, as n increases, and sketch the graph.

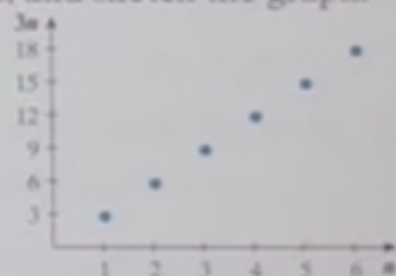
[Sol] Letting $a_n = 3n$,

$$a_1 = 3$$

$$a_2 = 6$$

$$a_3 = 9$$

⋮



As n increases, each term in the sequence **increases**. Therefore,

As n approaches infinity, the sequence **diverges to positive infinity**.

- Given that the general term of a sequence is 2^n , analyze how the terms of the sequence progress, as n increases, and sketch the graph.

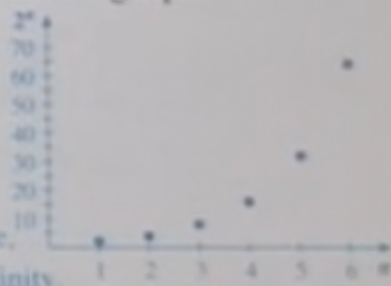
[Sol] Letting $a_n = 2^n$,

$$a_1 = 2$$

$$a_2 = 4$$

$$a_3 = 8$$

⋮



As n increases, each term in the sequence increases. Therefore,

As n approaches infinity, the sequence diverges to positive infinity.

Terminology and Notation: Infinity (denoted by the symbol ∞) is used to indicate a number that is greater than any fixed number. The notation $n \rightarrow \infty$ is read: "as n approaches infinity."

N 81 b

Given that the general term of a sequence is $-n + 3$, let's analyze how the terms of the sequence progress as $n \rightarrow \infty$.

Letting $a_n = -n + 3$,

$$a_1 = 2$$

$$a_2 = 1$$

$$a_3 = 0$$

$$a_4 = -1$$

$\vdots$



Note: As n increases, each term in the sequence decreases. Therefore, as $n \rightarrow \infty$, the sequence **diverges** to *negative infinity*.

3. Given that the general term of a sequence is $-2n$, analyze how the terms of the sequence progress, as $n \rightarrow \infty$, and sketch the graph.

[Sol] Letting $a_n = -2n$,

$$a_1 = -2$$

$$a_2 = -4$$

$$a_3 = -6$$

$\vdots$



As n increases, each term in the sequence decreases.

Therefore, as $n \rightarrow \infty$, the sequence diverges to negative infinity.

4. Given that the general term of a sequence is -3^n , analyze how the terms of the sequence progress, as $n \rightarrow \infty$, and sketch the graph.

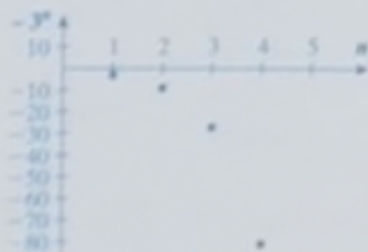
[Sol] Letting $a_n = -3^n$,

$$a_1 = -3$$

$$a_2 = -9$$

$$a_3 = -27$$

$\vdots$



As n increases, each term in the sequence decreases.

Therefore, as $n \rightarrow \infty$, the sequence diverges to negative infinity.

Infinite Sequences

Time : to : Date Name

100%	90%	80%	70%	60%~
(mistakes) 0	-	-	-	1-

Given that the general term of a sequence is $\frac{1}{n}$, let's analyze how the terms of the sequence progress as $n \rightarrow \infty$.

$$\text{Letting } a_n = \frac{1}{n},$$

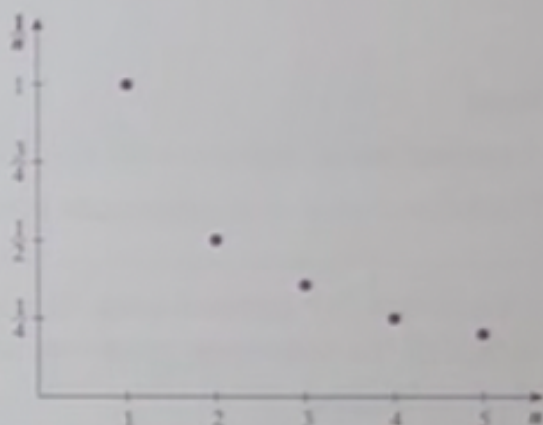
$$a_1 = 1$$

$$a_2 = \frac{1}{2}$$

$$a_3 = \frac{1}{3}$$

$$a_4 = \frac{1}{4}$$

⋮



Note:

As n increases, the terms of the sequence decrease but, never reach 0. Therefore, as $n \rightarrow \infty$, the sequence **converges** to a limit.

The limit value is 0.

- Given that the general term of a sequence is $\frac{2n+1}{n}$, analyze how the terms of the sequence progress, as n increases, and sketch the graph.

[Sol] Letting $a_n = \frac{2n+1}{n}$,

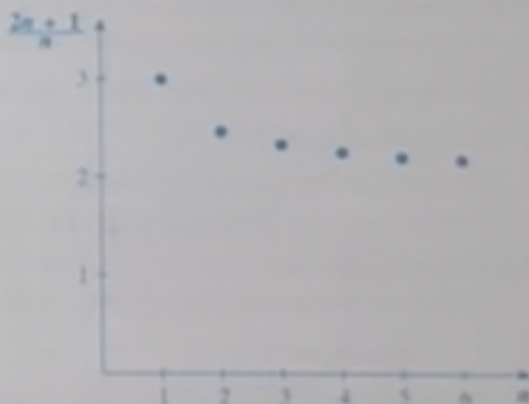
$$a_1 = 3$$

$$a_2 = \frac{5}{2} = 2\frac{1}{2}$$

$$a_3 = \frac{7}{3} = 2\frac{1}{3}$$

$$a_4 = \frac{9}{4} = 2\frac{1}{4}$$

⋮



As n increases, the terms of the sequence decrease but, never reach 2.

Therefore, as $n \rightarrow \infty$, the sequence converges to a limit. The limit value is 2.

Given that the general term of a sequence is $(-1)^n$, let's analyze how the terms of the sequence progress as $n \rightarrow \infty$.

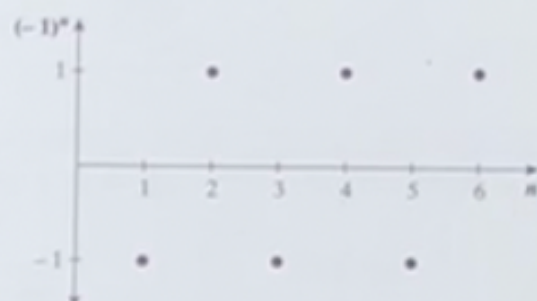
Letting $a_n = (-1)^n$,

$$a_1 = -1$$

$$a_2 = 1$$

$$a_3 = -1$$

$$a_4 = 1$$

$$\vdots$$


Note:

As n increases, the terms of the sequence do not converge to a limit. Therefore, as $n \rightarrow \infty$, the sequence **diverges**. This sequence **oscillates**.

2. Given that the general term of a sequence is $2 + (-1)^n$, analyze how the terms of the sequence progress, as n increases, and sketch the graph.

[Sol] Letting $a_n = 2 + (-1)^n$,

$$a_1 = 1$$

$$a_2 = 3$$

$$a_3 = 1$$

$$a_4 = 3$$

$$\vdots$$


As n increases, the terms of the sequence do not converge to a limit. Therefore, as $n \rightarrow \infty$, the sequence diverges. This sequence oscillates.

Note Summary:

In Worksheets N31-80 we studied sequences with finite terms, called *Finite Sequences*. We will now begin to study sequences whose terms continue infinitely, and are called *Infinite Sequences*.

Given the infinite sequence $\{a_n\}$: $a_1, a_2, a_3, \dots, a_n, \dots$,

As the value of n increases:

- When a_n approaches a constant value L , $\{a_n\}$ is said to **converge** to L , where L is the limit value and is expressed as: $\lim_{n \rightarrow \infty} a_n = L$.
- When a_n **diverges** to positive (or negative) infinity, the limit is positive (or negative) infinity, and is expressed as:

$$\lim_{n \rightarrow \infty} a_n = +\infty$$

$$\lim_{n \rightarrow \infty} a_n = -\infty$$

Divergent sequences which do not diverge to positive or negative infinity are said to **oscillate**.

Infinite Sequences

Time : to : Date Name

100%	90%	80%	70%	60%~
(mistakes) 0	1	2-3	4	5-

1. In each of the following exercises, state whether the given sequence converges or diverges. If it converges, state the limit value. If it does not converge to a limit value, write "*diverges*".

Ex. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots$ (0)

(1) $1, 2, 4, 8, 16, \dots$ (*diverges*)

(2) $99, 97, 95, 93, 91, \dots$ (*diverges*)

(3) $\frac{1}{1}, \frac{4}{3}, \frac{9}{5}, \frac{16}{7}, \frac{25}{9}, \dots$ (*diverges*)

(4) $\frac{4}{1}, \frac{7}{2}, \frac{10}{3}, \frac{13}{4}, \frac{16}{5}, \dots$ (3)

(5) $2, 4, 2, 4, 2, \dots$ (*diverges*)

(6) $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \dots$ (0)

(7) $\frac{2}{1}, \frac{5}{2}, \frac{10}{3}, \frac{17}{4}, \frac{26}{5}, \dots$ (*diverges*)

(8) $5, 7, 5, 7, 5, \dots$ (*diverges*)

N 83 b

2. Determine the general terms of the sequences from side a, exercise 1. (1)-(8), and express their limits.

Ex.

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

$$(1) \lim_{n \rightarrow \infty} 2^{n-1} = \infty$$

$$(2) \lim_{n \rightarrow \infty} [100 - (2n - 1)] = \lim_{n \rightarrow \infty} (101 - 2n) = -\infty$$

$$(3) \lim_{n \rightarrow \infty} \frac{n^2}{2n-1} = \lim_{n \rightarrow \infty} \frac{n}{2 - \frac{1}{n}} = \infty$$

$$(4) \lim_{n \rightarrow \infty} \frac{3n+1}{n} = \lim_{n \rightarrow \infty} \left(3 + \frac{1}{n} \right) = 3$$

$$(5) \lim_{n \rightarrow \infty} [3 + (-1)^n] \quad \text{The sequence oscillates.}$$

$$(6) \lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 0$$

$$(7) \lim_{n \rightarrow \infty} \frac{n^2+1}{n} = \lim_{n \rightarrow \infty} \left(n + \frac{1}{n} \right) = \infty$$

$$(8) \lim_{n \rightarrow \infty} [6 + (-1)^n] \quad \text{The sequence oscillates.}$$

Infinite Sequences

Time : to : Date Name

100%	90%	80%	70%	60%~
(mistakes) 0	1	2	3	4

Infinite Sequences are classified as follows:

Infinite Sequences	{	Converge	--- ①
		Diverge	{ to positive infinity to negative infinity oscillate
			--- ②
			--- ③
			--- ④

In each of the following exercises, classify the infinite sequence of the given general term, by indicating the appropriate number ①-④. If the sequence converges, state the limit value.

Ex.

$2n + 5$

(②)

(1) $\frac{1}{n^2 + 1}$

(①, limit value 0)

(2) $-2n + 5$

(③)

(3) $\frac{2n + 1}{n}$

(①, limit value 2)

(4) $(-3)^n$

(④)

N 84 b

(5) $2n - 1$

(②)

(6) $\frac{n+1}{n^2}$

(①, limit value 0)

(7) $\frac{n^2+1}{n^2}$

(①, limit value 1)

(8) $\frac{n^2-1}{n}$

(②)

(9) $1 + (-1)^n \frac{1}{n}$

(①, limit value 1)

(10) $\frac{3-2n}{5+4n}$

(①, limit value $-\frac{1}{2}$)

(11) $\frac{n^2-n-2}{2n^2+1}$

(①, limit value $\frac{1}{2}$)

Infinite Sequences

Time : to : Date Name

100%	90%	80%	70%	60%~
(mistakes) 0	1	2	3	4-

In each of the following exercises, evaluate the given limit.

Ex. $\lim_{n \rightarrow \infty} \frac{2n-1}{3n+1}$

[Sol] Dividing both the numerator and the denominator by the highest powered variable of the denominator, in this case n ,

$$\lim_{n \rightarrow \infty} \frac{2n-1}{3n+1} = \lim_{n \rightarrow \infty} \frac{2 - \frac{1}{n}}{3 + \frac{1}{n}}$$

As $n \rightarrow \infty$, $2 - \frac{1}{n} \rightarrow 2$ and $3 + \frac{1}{n} \rightarrow 3$

$$\therefore \lim_{n \rightarrow \infty} \frac{2n-1}{3n+1} = \lim_{n \rightarrow \infty} \frac{2 - \frac{1}{n}}{3 + \frac{1}{n}} = \frac{2}{3}$$

$$(1) \lim_{n \rightarrow \infty} \frac{2n-3}{4n+5} = \lim_{n \rightarrow \infty} \frac{2 - \frac{3}{n}}{4 + \frac{5}{n}} = \frac{2}{4}$$

$$(2) \lim_{n \rightarrow \infty} \frac{5n-8}{n^2+4} = \lim_{n \rightarrow \infty} \frac{\frac{5}{n} - \frac{8}{n^2}}{1 + \frac{4}{n^2}} = 0$$

$$(3) \lim_{n \rightarrow \infty} \frac{2n^2-1}{3n^2+4} = \lim_{n \rightarrow \infty} \frac{2 - \frac{1}{n^2}}{3 + \frac{4}{n^2}} = \frac{2}{3}$$

$$(4) \lim_{n \rightarrow \infty} \frac{n^2-n-2}{2n^2+1} = \lim_{n \rightarrow \infty} \frac{1 - \frac{1}{n} - \frac{2}{n^2}}{2 + \frac{1}{n^2}} = \frac{1}{2}$$

(Note: The highest powered variable of the denominator is n^2 .)

Answers: $\frac{1}{4}, \frac{2}{3}, \frac{1}{2}, \frac{1}{2}$

N 85 b

$$(5) \lim_{n \rightarrow \infty} \frac{11 - 2n}{6n + 4} = \lim_{n \rightarrow \infty} \frac{\frac{11}{n} - 2}{6 + \frac{4}{n}} = -\frac{1}{3}$$

$$(6) \lim_{n \rightarrow \infty} \frac{n^2 + 5n + 4}{3 - 2n^2} = \lim_{n \rightarrow \infty} \frac{1 + \frac{5}{n} + \frac{4}{n^2}}{\frac{3}{n^2} - 2} = -\frac{1}{2}$$

$$(7) \lim_{n \rightarrow \infty} \frac{n^3 + n + 4}{2 - 3n^3} = \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{n^2} + \frac{4}{n^3}}{\frac{2}{n^3} - 3} = -\frac{1}{3}$$

$$(8) \lim_{n \rightarrow \infty} \frac{4n^2 + 1}{n^3 - 2n^2} = \lim_{n \rightarrow \infty} \frac{\frac{4}{n} + \frac{1}{n^3}}{1 - \frac{2}{n}} = 0$$

$$(9) \lim_{n \rightarrow \infty} \frac{-2n^3 - 4n^2}{2n^2 - 5} = \lim_{n \rightarrow \infty} \frac{-2n - 4}{2 - \frac{5}{n^2}} = -\infty$$

$$(10) \lim_{n \rightarrow \infty} \frac{-2n^3 + 5n}{n^4 + 2n + 1} = \lim_{n \rightarrow \infty} \frac{-\frac{2}{n} + \frac{5}{n^3}}{1 + \frac{2}{n^3} + \frac{1}{n^4}} = 0$$

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	1	2	3	4

Evaluate each of the following limits.

Ex. $\lim_{n \rightarrow \infty} \frac{(n-1)(3n-2)}{n^2}$

$= \lim_{n \rightarrow \infty} \left[\left(\frac{n-1}{n} \right) \left(\frac{3n-2}{n} \right) \right]$

$= \lim_{n \rightarrow \infty} \left[\left(1 - \frac{1}{n} \right) \left(3 - \frac{2}{n} \right) \right]$

$= 3$

Separate into the product of two fractions.

Divide both numerators and denominators by n .

As $n \rightarrow \infty$, $1 - \frac{1}{n} \rightarrow 1$ and $3 - \frac{2}{n} \rightarrow 3$.

$$(1) \lim_{n \rightarrow \infty} \frac{(2n+1)(2n-1)}{4n^2} = \lim_{n \rightarrow \infty} \left[\left(\frac{2n+1}{2n} \right) \left(\frac{2n-1}{2n} \right) \right]$$

$$= \lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{2n} \right) \left(1 - \frac{1}{2n} \right) \right]$$

$$= 1$$

$$(2) \lim_{n \rightarrow \infty} \frac{n-2}{3n(n+2)} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n} - \frac{2}{n^2}}{3 \left(1 + \frac{2}{n} \right)}$$

$$= 0$$

$$(3) \lim_{n \rightarrow \infty} \frac{n-7n^3}{5n^2-n+3} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n} - 7n}{5 - \frac{1}{n} + \frac{3}{n^2}}$$

$$= -\infty$$

$$(4) \lim_{n \rightarrow \infty} \frac{10-4n^2+6n^4}{5n^4-2n^3+n^2-3n} = \lim_{n \rightarrow \infty} \frac{\frac{10}{n^4} - \frac{4}{n^2} + 6}{5 - \frac{2}{n} + \frac{1}{n^2} - \frac{3}{n^3}} = \frac{6}{5}$$

N 86 b

$$(5) \lim_{n \rightarrow \infty} (\sqrt{n} - 100) = \infty$$

$$(6) \lim_{n \rightarrow \infty} \frac{100}{\sqrt{n}} = 0$$

$$(7) \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\sqrt{n} + 5} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{5}{\sqrt{n}}} = 1$$

$$(8) \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{n} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$$

Note: $\frac{\sqrt{n}}{n} = \frac{n^{\frac{1}{2}}}{n^1} = n^{\frac{1}{2} - 1} = n^{-\frac{1}{2}} = \frac{1}{\sqrt{n}}$

$$(9) \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{n+1} = \lim_{n \rightarrow \infty} \frac{\frac{1}{\sqrt{n}}}{1 + \frac{1}{n}} = 0$$

$$(10) \lim_{n \rightarrow \infty} \frac{-3n + \sqrt{n}}{n-1} = \lim_{n \rightarrow \infty} \frac{-3 + \frac{1}{\sqrt{n}}}{1 - \frac{1}{n}} = -3$$

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	1	-	2

Evaluate each of the following limits.

Ex.

$$\begin{aligned}
 & \lim_{n \rightarrow \infty} (\sqrt{n+1} - \sqrt{n}) \\
 &= \lim_{n \rightarrow \infty} \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} \\
 &= \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} \\
 &= 0
 \end{aligned}$$

$$(1) \lim_{n \rightarrow \infty} (\sqrt{n^2 + n + 1} - n)$$

$$= \lim_{n \rightarrow \infty} \frac{(\sqrt{n^2 + n + 1} - n)(\sqrt{n^2 + n + 1} + n)}{\sqrt{n^2 + n + 1} + n}$$

$$= \lim_{n \rightarrow \infty} \frac{n+1}{\sqrt{n^2 + n + 1} + n}$$

$$= \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{n}}{\sqrt{1 + \frac{1}{n} + \frac{1}{n^2}} + 1} = \frac{1}{2}$$

$$(2) \lim_{n \rightarrow \infty} (\sqrt{n^2 - 4n} - n)$$

$$= \lim_{n \rightarrow \infty} \frac{(\sqrt{n^2 - 4n} - n)(\sqrt{n^2 - 4n} + n)}{\sqrt{n^2 - 4n} + n}$$

$$= \lim_{n \rightarrow \infty} \frac{-4n}{\sqrt{n^2 - 4n} + n}$$

$$= \lim_{n \rightarrow \infty} \frac{-4}{\sqrt{1 - \frac{4}{n}} + 1} = -2$$

N 87 b

$$(3) \lim_{n \rightarrow \infty} (\sqrt{n+1} - \sqrt{n-1})$$

$$= \lim_{n \rightarrow \infty} \frac{(\sqrt{n+1} - \sqrt{n-1})(\sqrt{n+1} + \sqrt{n-1})}{\sqrt{n+1} + \sqrt{n-1}}$$

$$= \lim_{n \rightarrow \infty} \frac{2}{\sqrt{n+1} + \sqrt{n-1}}$$

$$= 0$$

$$(4) \lim_{n \rightarrow \infty} (\sqrt{n^2 - n + 1} - n)$$

$$= \lim_{n \rightarrow \infty} \frac{(\sqrt{n^2 - n + 1} - n)(\sqrt{n^2 - n + 1} + n)}{\sqrt{n^2 - n + 1} + n}$$

$$= \lim_{n \rightarrow \infty} \frac{-n + 1}{\sqrt{n^2 - n + 1} + n}$$

$$= \lim_{n \rightarrow \infty} \frac{-1 + \frac{1}{n}}{\sqrt{1 - \frac{1}{n} + \frac{1}{n^2}} + 1}$$

$$= -\frac{1}{2}$$

$$(5) \lim_{n \rightarrow \infty} (\sqrt{n^2 - 5n - 2} - n)$$

$$= \lim_{n \rightarrow \infty} \frac{(\sqrt{n^2 - 5n - 2} - n)(\sqrt{n^2 - 5n - 2} + n)}{\sqrt{n^2 - 5n - 2} + n}$$

$$= \lim_{n \rightarrow \infty} \frac{-5n - 2}{\sqrt{n^2 - 5n - 2} + n}$$

$$= \lim_{n \rightarrow \infty} \frac{-5 - \frac{2}{n}}{\sqrt{1 - \frac{5}{n} - \frac{2}{n^2}} + 1}$$

$$= -\frac{5}{2}$$

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	-	1	-	2-

Evaluate each of the following limits.

$$(1) \lim_{n \rightarrow \infty} (\sqrt{2n+1} - \sqrt{3n})$$

$$= \lim_{n \rightarrow \infty} \frac{(\sqrt{2n+1} - \sqrt{3n})(\sqrt{2n+1} + \sqrt{3n})}{\sqrt{2n+1} + \sqrt{3n}}$$

$$= \lim_{n \rightarrow \infty} \frac{-n+1}{\sqrt{2n+1} + \sqrt{3n}}$$

$$= \lim_{n \rightarrow \infty} \frac{-\sqrt{n} + \frac{1}{\sqrt{n}}}{\sqrt{2 + \frac{1}{n}} + \sqrt{3}}$$

$$= -\infty$$

$$(2) \lim_{n \rightarrow \infty} \sqrt{n}(\sqrt{n+1} - \sqrt{n})$$

$$= \lim_{n \rightarrow \infty} \frac{\sqrt{n}(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1 + \frac{1}{n}} + 1}$$

$$= \frac{1}{2}$$

$$(3) \lim_{n \rightarrow \infty} \frac{\sqrt{n^2 + n} - n}{n}$$

$$= \lim_{n \rightarrow \infty} \left(\sqrt{1 + \frac{1}{n}} - 1 \right)$$

$$= 0$$

N 88 b

$$\begin{aligned}
 (4) \quad \lim_{n \rightarrow \infty} \frac{\sqrt{n+5} - \sqrt{n+3}}{\sqrt{n+1} - \sqrt{n}} &= \lim_{n \rightarrow \infty} \frac{(\sqrt{n+5} - \sqrt{n+3})(\sqrt{n+5} + \sqrt{n+3})(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+5} + \sqrt{n+3})(\sqrt{n+1} + \sqrt{n})} \\
 &= \lim_{n \rightarrow \infty} \frac{2(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+5} + \sqrt{n+3}} \\
 &= \lim_{n \rightarrow \infty} \frac{2\left(\sqrt{1 + \frac{1}{n}} + 1\right)}{\sqrt{1 + \frac{5}{n}} + \sqrt{1 + \frac{3}{n}}} \\
 &= 2
 \end{aligned}$$

$$\begin{aligned}
 (5) \quad \lim_{n \rightarrow \infty} \frac{\sqrt{2n+1} - \sqrt{2n-1}}{\sqrt{4n+1} - 2\sqrt{n}} &= \lim_{n \rightarrow \infty} \frac{2(\sqrt{4n+1} + 2\sqrt{n})}{\sqrt{2n+1} + \sqrt{2n-1}} \\
 &= \lim_{n \rightarrow \infty} \frac{2\left(\sqrt{4 + \frac{1}{n}} + 2\right)}{\sqrt{2 + \frac{1}{n}} + \sqrt{2 - \frac{1}{n}}} \\
 &= 2\sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 (6) \quad \lim_{n \rightarrow \infty} \frac{\sqrt{n} - \sqrt{n+2}}{\sqrt{n^2+5} - n} &= \lim_{n \rightarrow \infty} \frac{-2(\sqrt{n^2+5} + n)}{5(\sqrt{n} + \sqrt{n+2})} \\
 &= \lim_{n \rightarrow \infty} \frac{-2\left(\sqrt{n + \frac{5}{n}} + \sqrt{n}\right)}{5\left(1 + \sqrt{1 + \frac{2}{n}}\right)}
 \end{aligned}$$

Infinite Sequences

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	-	1	2

1. Evaluate each of the following limits.

Ex. $\lim_{n \rightarrow \infty} \frac{1 + 2 + \dots + n}{n^2}$

$= \lim_{n \rightarrow \infty} \frac{\frac{n(n+1)}{2}}{n^2}$

$= \lim_{n \rightarrow \infty} \frac{n+1}{2n}$

$= \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{n}}{2}$

$= \frac{1}{2}$

Express the numerator as the sum of an arithmetic sequence.

Divide the numerator and denominator by n .

Answer: $\frac{n(n+1)}{2}, 1 + \frac{1}{n}, \frac{1}{2}$

$$\begin{aligned}
 (1) \quad & \lim_{n \rightarrow \infty} \frac{1^2 + 2^2 + \dots + n^2}{n^3} \\
 &= \lim_{n \rightarrow \infty} \frac{\frac{n(n+1)(2n+1)}{6}}{n^3} \\
 &= \lim_{n \rightarrow \infty} \frac{(n+1)(2n+1)}{6n^2} \\
 &= \lim_{n \rightarrow \infty} \left[\frac{1}{6} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) \right] \\
 &= \frac{1}{3}
 \end{aligned}$$

N 89 b

2. Evaluate each of the following limits.

$$(1) \lim_{n \rightarrow \infty} \frac{n\sqrt{n}}{n^2 - 1}$$

$$= \lim_{n \rightarrow \infty} \frac{\frac{1}{\sqrt{n}}}{1 - \frac{1}{n^2}}$$

$$= 0$$

$$(2) \lim_{n \rightarrow \infty} (\sqrt{n^2 + 8n + 2} - \sqrt{n^2 + 2n + 3})$$

$$= \lim_{n \rightarrow \infty} \frac{(\sqrt{n^2 + 8n + 2} - \sqrt{n^2 + 2n + 3})(\sqrt{n^2 + 8n + 2} + \sqrt{n^2 + 2n + 3})}{\sqrt{n^2 + 8n + 2} + \sqrt{n^2 + 2n + 3}}$$

$$= \lim_{n \rightarrow \infty} \frac{6n - 1}{\sqrt{n^2 + 8n + 2} + \sqrt{n^2 + 2n + 3}}$$

$$= \lim_{n \rightarrow \infty} \frac{6 - \frac{1}{n}}{\sqrt{1 + \frac{8}{n} + \frac{2}{n^2}} + \sqrt{1 + \frac{2}{n} + \frac{3}{n^2}}}$$

$$= 3$$

$$(3) \lim_{n \rightarrow \infty} (\sqrt{n^3 + n} - n^{\frac{3}{2}})$$

$$= \lim_{n \rightarrow \infty} \frac{(\sqrt{n^3 + n} - \sqrt{n^3})(\sqrt{n^3 + n} + \sqrt{n^3})}{\sqrt{n^3 + n} + \sqrt{n^3}}$$

$$= \lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^3 + n} + \sqrt{n^3}}$$

$$= \lim_{n \rightarrow \infty} \frac{\frac{1}{\sqrt{n}}}{\sqrt{1 + \frac{1}{n^2}} + 1}$$

$$= 0$$

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	1	-	2-

Evaluate each of the following limits.

$$(1) \lim_{n \rightarrow \infty} \frac{(n+1)(n+2)}{n^2}$$

$$= \lim_{n \rightarrow \infty} \left[\left(\frac{n+1}{n} \right) \left(\frac{n+2}{n} \right) \right]$$

$$= \lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{n} \right) \left(1 + \frac{2}{n} \right) \right]$$

$$= 1$$

$$(2) \lim_{n \rightarrow \infty} \frac{(n+1)(n-5)}{n(n-1)^2}$$

$$= \lim_{n \rightarrow \infty} \frac{\left(1 + \frac{1}{n} \right) \left(1 - \frac{5}{n} \right)}{n \left(1 - \frac{1}{n} \right)^2}$$

$$= 0$$

$$(3) \lim_{n \rightarrow \infty} (\sqrt{n^2 - 9n} - n)$$

$$= \lim_{n \rightarrow \infty} \frac{(\sqrt{n^2 - 9n} - n)(\sqrt{n^2 - 9n} + n)}{\sqrt{n^2 - 9n} + n}$$

$$= \lim_{n \rightarrow \infty} \frac{-9n}{\sqrt{n^2 - 9n} + n}$$

$$= \lim_{n \rightarrow \infty} \frac{-9}{\sqrt{1 - \frac{9}{n}} + 1}$$

$$= -\frac{9}{2}$$

N 90 b

$$(4) \lim_{n \rightarrow \infty} \frac{\sqrt{3n}}{\sqrt{2n+1}}$$

$$= \lim_{n \rightarrow \infty} \frac{\sqrt{3}}{\sqrt{2 + \frac{1}{n}}}$$

$$= \frac{\sqrt{6}}{2}$$

$$(5) \lim_{n \rightarrow \infty} (\sqrt{n^2 + n - 3} - \sqrt{n^2 - 3n + 4})$$

$$= \lim_{n \rightarrow \infty} \frac{(\sqrt{n^2 + n - 3} - \sqrt{n^2 - 3n + 4})(\sqrt{n^2 + n - 3} + \sqrt{n^2 - 3n + 4})}{\sqrt{n^2 + n - 3} + \sqrt{n^2 - 3n + 4}}$$

$$= \lim_{n \rightarrow \infty} \frac{4n - 7}{\sqrt{n^2 + n - 3} + \sqrt{n^2 - 3n + 4}}$$

$$= \lim_{n \rightarrow \infty} \frac{4 - \frac{7}{n}}{\sqrt{1 + \frac{1}{n} - \frac{3}{n^2}} + \sqrt{1 - \frac{3}{n} + \frac{4}{n^2}}}$$

$$= 2$$

$$(6) \lim_{n \rightarrow \infty} \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{2n+1} - \sqrt{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{\sqrt{1 + \frac{1}{n}} + 1}{\sqrt{2 + \frac{1}{n}} - 1}$$

$$= \frac{2}{\sqrt{2} - 1}$$

$$= 2(\sqrt{2} + 1)$$

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	1	2	3	4

In each of the following exercises, given the value of r , find the limit of the infinite geometric sequence $r, r^2, r^3, \dots, r^n, \dots$.

Ex. $r = \frac{1}{2}$

[Sol] The general term is $a_n = \left(\frac{1}{2}\right)^n$.

When $n \rightarrow \infty$, $a_n \rightarrow 0$.

Therefore, the sequence converges to 0.

(1) $r = \frac{1}{3}$

[Sol] The general term is $a_n = \left(\frac{1}{3}\right)^n$.

When $n \rightarrow \infty$, $a_n \rightarrow 0$. Therefore, the sequence converges to 0.

(2) $r = 2$

[Sol] The general term is $a_n = 2^n$. When $n \rightarrow \infty$, $a_n \rightarrow \infty$.

Therefore, the sequence diverges to positive infinity.

(3) $r = -3$

[Sol] The general term is $a_n = (-3)^n$.

Therefore, the sequence oscillates.

(4) $r = 1$

[Sol] The general term is $a_n = 1^n$.

When $n \rightarrow \infty$, $a_n \rightarrow 1$. Therefore, the sequence converges to 1.

(5) $r = -1$

[Sol] The general term is $a_n = (-1)^n$.

Therefore, the sequence oscillates.

(6) $r = -\frac{2}{3}$

[Sol] The general term is $a_n = \left(-\frac{2}{3}\right)^n$.

When $n \rightarrow \infty$, $a_n \rightarrow 0$. Therefore, the sequence converges to 0.

N 91 b

(7) $r = -2$

[Sol] The general term is $a_n = (-2)^n$.

Therefore, the sequence oscillates.

(8) $r = -0.5$

[Sol] The general term is $a_n = \left(-\frac{1}{2}\right)^n$.

When $n \rightarrow \infty$, $a_n \rightarrow 0$.

Therefore, the sequence converges to 0.

(9) $r = 5$

[Sol] The general term is $a_n = 5^n$.

When $n \rightarrow \infty$, $a_n \rightarrow \infty$.

Therefore, the sequence diverges to positive infinity.

(10) $r = \frac{1}{10}$

[Sol] The general term is $a_n = \left(\frac{1}{10}\right)^n$.

When $n \rightarrow \infty$, $a_n \rightarrow 0$.

Therefore, the sequence converges to 0.

(11) $r = -\frac{1}{3}$

[Sol] The general term is $a_n = \left(-\frac{1}{3}\right)^n$.

When $n \rightarrow \infty$, $a_n \rightarrow 0$.

Therefore, the sequence converges to 0.

(12) $r = -5$

[Sol] The general term is $a_n = (-5)^n$.

Therefore, the sequence oscillates.

Infinite Geometric Sequences 1

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	1	2	3-4	5-

Given the infinite geometric sequence $r, r^2, r^3, \dots, r^n, \dots$

When $r > 1$,

$$\lim_{n \rightarrow \infty} r^n = +\infty$$

When $r = 1$,

$$\lim_{n \rightarrow \infty} r^n = 1$$

When $-1 < r < 1$,

$$\lim_{n \rightarrow \infty} r^n = 0$$

When $r \leq -1$,

The sequence oscillates.

In each of the following exercises, find the limit of the infinite geometric sequence of the given general term.

Ex. $\left(-\frac{4}{5}\right)^n$ The limit value is 0.

(1) $\left(-\frac{8}{9}\right)^n$ The limit value is 0.

(2) $\left(\frac{8}{9}\right)^n$ The limit value is 0.

(3) $\left(\frac{\sqrt{2}}{3}\right)^n$ The limit value is 0.

(4) $\left(\frac{5}{3}\right)^n$ The sequence diverges to $+\infty$.

(5) $\left(-\frac{13}{2}\right)^n$ The sequence oscillates.

(6) $\left(-\frac{4}{\sqrt{3}}\right)^n$ The sequence oscillates.

N 92 b

Ex. $1 + \left(-\frac{1}{3}\right)^n$

The limit value is 1.

(7) $1 + \left(-\frac{7}{3}\right)^n$

The sequence oscillates.

(8) $1 + 5^n$

The sequence diverges to $+\infty$.

(9) $1 + \left(\frac{5}{8}\right)^n$

The limit value is 1.

(10) $1 + (-1)^n$

The sequence oscillates.

(11) $2\left(\frac{2}{\sqrt{3}}\right)^n$

The sequence diverges to $+\infty$.

(12) $\frac{2^n}{(\sqrt{5})^n}$

The limit value is 0.

(13) $\left(\frac{1}{2}\right)^n + (-2)^n$

The sequence oscillates.

(14) $\frac{4}{3}\left(\frac{3}{4}\right)^{n-1}$

The limit value is 0.

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	—	1	2	3

Evaluate the following limits.

Ex.

$$\lim_{n \rightarrow \infty} \frac{3^n}{4^n + 1}$$

$$= \lim_{n \rightarrow \infty} \frac{\left(\frac{3}{4}\right)^n}{1 + \left(\frac{1}{4}\right)^n}$$

$$= 0$$

Divide both the numerator and the denominator by 4^n .

$$(1) \lim_{n \rightarrow \infty} \frac{2^n}{5^n + 1} = \lim_{n \rightarrow \infty} \frac{\left(\frac{2}{5}\right)^n}{1 + \left(\frac{1}{5}\right)^n} = 0$$

$$(2) \lim_{n \rightarrow \infty} \frac{1 + 2^n}{1 + 3^n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{3}\right)^n + \left(\frac{2}{3}\right)^n}{\left(\frac{1}{3}\right)^n + 1} = 0$$

$$(3) \lim_{n \rightarrow \infty} \frac{3^n + 2^n}{2^{2n} - 3^n} = \lim_{n \rightarrow \infty} \frac{3^n + 2^n}{4^n - 3^n}$$

$$= \lim_{n \rightarrow \infty} \frac{\left(\frac{3}{4}\right)^n + \left(\frac{1}{2}\right)^n}{1 - \left(\frac{3}{4}\right)^n}$$

$$= 0$$

N 93 b

$$\begin{aligned}(4) \quad \lim_{n \rightarrow \infty} \frac{5^n + 4^n}{3^{2n} - 2^n} &= \lim_{n \rightarrow \infty} \frac{5^n + 4^n}{9^n - 2^n} \\&= \lim_{n \rightarrow \infty} \frac{\left(\frac{5}{9}\right)^n + \left(\frac{4}{9}\right)^n}{1 - \left(\frac{2}{9}\right)^n} \\&= 0\end{aligned}$$

$$\begin{aligned}(5) \quad \lim_{n \rightarrow \infty} \frac{2^{2n+1}}{4^n - 1} &= \lim_{n \rightarrow \infty} \frac{2^{2n} \cdot \boxed{2}}{4^n - 1} \\&= \lim_{n \rightarrow \infty} \frac{4^n \cdot 2}{4^n - 1} \\&= \lim_{n \rightarrow \infty} \frac{2}{1 - \left(\frac{1}{4}\right)^n} \\&= 2\end{aligned}$$

$$\begin{aligned}(6) \quad \lim_{n \rightarrow \infty} \frac{3^{2n+1}}{9^n - 1} &= \lim_{n \rightarrow \infty} \frac{3^{2n} \cdot 3}{9^n - 1} \\&= \lim_{n \rightarrow \infty} \frac{9^n \cdot 3}{9^n - 1} \\&= \lim_{n \rightarrow \infty} \frac{3}{1 - \left(\frac{1}{9}\right)^n} \\&= 3\end{aligned}$$

$$\begin{aligned}(7) \quad \lim_{n \rightarrow \infty} \frac{5^{2n+3}}{25^n - 1} &= \lim_{n \rightarrow \infty} \frac{5^{2n} \cdot 5^3}{25^n - 1} \\&= \lim_{n \rightarrow \infty} \frac{25^n \cdot 125}{25^n - 1} \\&= \lim_{n \rightarrow \infty} \frac{125}{1 - \left(\frac{1}{25}\right)^n} \\&= 125\end{aligned}$$

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	1	2	3	4

Evaluate the following limits.

$$(1) \lim_{n \rightarrow \infty} \frac{4^n - 2^n}{3^n} = \lim_{n \rightarrow \infty} \left[\left(\frac{4}{3}\right)^n - \left(\frac{2}{3}\right)^n \right]$$

$$= \infty$$

$$(2) \lim_{n \rightarrow \infty} \frac{6^n - 1^n}{3^n} = \lim_{n \rightarrow \infty} \left[2^n - \left(\frac{1}{3}\right)^n \right]$$

$$= \infty$$

$$(3) \lim_{n \rightarrow \infty} \frac{3^n + 2^n}{4^n} = \lim_{n \rightarrow \infty} \left[\left(\frac{3}{4}\right)^n + \left(\frac{1}{2}\right)^n \right] = 0$$

$$(4) \lim_{n \rightarrow \infty} \frac{5^n + 6^n}{6^n} = \lim_{n \rightarrow \infty} \left[\left(\frac{5}{6}\right)^n + 1 \right] = 1$$

$$(5) \lim_{n \rightarrow \infty} \frac{2^n + 8^n}{4^n} = \lim_{n \rightarrow \infty} \left[\left(\frac{1}{2}\right)^n + 2^n \right]$$

$$= \infty$$

N 94 b

$$\begin{aligned}
 (6) \quad \lim_{n \rightarrow \infty} \frac{4^{n+1} + 3^n}{2^{2n} - (-3)^{n+1}} &= \lim_{n \rightarrow \infty} \frac{4^n \cdot 4 + 3^n}{4^n - (-3) \cdot (-3)^n} \\
 &= \lim_{n \rightarrow \infty} \frac{4 + \left(\frac{3}{4}\right)^n}{1 + 3\left(-\frac{3}{4}\right)^n} = 4
 \end{aligned}$$

$$\begin{aligned}
 (7) \quad \lim_{n \rightarrow \infty} \frac{9^{n+1} + 2^n}{3^{2n} - (-2)^{n+1}} &= \lim_{n \rightarrow \infty} \frac{9^n \cdot 9 + 2^n}{9^n - (-2) \cdot (-2)^n} \\
 &= \lim_{n \rightarrow \infty} \frac{9 + \left(\frac{2}{9}\right)^n}{1 + 2\left(-\frac{2}{9}\right)^n} = 9
 \end{aligned}$$

$$\begin{aligned}
 (8) \quad \lim_{n \rightarrow \infty} \frac{4^{n+2} + 2^n}{2^{2n} + 4^n} &= \lim_{n \rightarrow \infty} \frac{4^n \cdot 16 + 2^n}{2 \cdot 4^n} \\
 &= \lim_{n \rightarrow \infty} \frac{16 + \left(\frac{1}{2}\right)^n}{2} \\
 &= 8
 \end{aligned}$$

$$(9) \quad \lim_{n \rightarrow \infty} (3^n - 2^n) = \lim_{n \rightarrow \infty} \left\{ 3^n \left[1 - \left(\frac{2}{3}\right)^n \right] \right\} = \infty$$

$$(10) \quad \lim_{n \rightarrow \infty} (7^n + 9^n) = \lim_{n \rightarrow \infty} \left\{ 9^n \left[\left(\frac{7}{9}\right)^n + 1 \right] \right\} = \infty$$

Infinite Geometric Sequences 1

Time : to : Date Name

100% (mistakes) 0	90%	80%	70%	69%~
-	1	2	3-	

In each of the following exercises, determine if the sequence whose general term is given converges or diverges. If it converges, state the limit value.

(1) $\frac{2^n + 6^n}{3^n}$ [Sol] The general term is $a_n = \frac{2^n + 6^n}{3^n} = \left(\frac{2}{3}\right)^n + \boxed{2^n}$.

When $n \rightarrow \infty$, $a_n \rightarrow \boxed{\infty}$.

Therefore, the sequence diverges to $+\infty$.

(2) $\frac{3^n + 5^n}{4^n + 5^n}$ [Sol] The general term is $a_n = \frac{3^n + 5^n}{4^n + 5^n} = \frac{\left(\frac{3}{5}\right)^n + 1}{\left(\frac{4}{5}\right)^n + 1}$.

When $n \rightarrow \infty$, $a_n \rightarrow 1$.

Therefore, the sequence converges to 1.

(3) $\frac{5^n}{6^n + 1}$ [Sol] The general term is $a_n = \frac{5^n}{6^n + 1} = \frac{\left(\frac{5}{6}\right)^n}{1 + \left(\frac{1}{6}\right)^n}$.

When $n \rightarrow \infty$, $a_n \rightarrow 0$.

Therefore, the sequence converges to 0.

(4) $\frac{(-3)^n}{2^n - 1}$ [Sol] The general term is $a_n = \frac{(-3)^n}{2^n - 1} = \frac{\left(-\frac{3}{2}\right)^n}{1 - \left(\frac{1}{2}\right)^n}$.

Therefore, the sequence diverges and oscillates.

N 95 b

(5) $\frac{(-3)^n - 1}{2^{2n}}$ [Sol] The general term is:

$$a_n = \frac{(-3)^n - 1}{2^{2n}} = \frac{(-3)^n - 1}{4^n} = \left(-\frac{3}{4}\right)^n - \left(\frac{1}{4}\right)^n.$$

When $n \rightarrow \infty$, $a_n \rightarrow 0$.

Therefore, the sequence converges to 0.

(6) $\frac{4^{n+1} + 3^n}{2^{2n}}$ [Sol] The general term is:

$$a_n = \frac{4^{n+1} + 3^n}{2^{2n}} = \frac{4^n \cdot 4 + 3^n}{4^n} = 4 + \left(\frac{3}{4}\right)^n.$$

When $n \rightarrow \infty$, $a_n \rightarrow 4$.

Therefore, the sequence converges to 4.

(7) $\frac{\sqrt{5^n} + 1}{2^n}$ [Sol] The general term is:

$$a_n = \frac{\sqrt{5^n} + 1}{2^n} = \left(\frac{\sqrt{5}}{2}\right)^n + \left(\frac{1}{2}\right)^n.$$

When $n \rightarrow \infty$, $a_n \rightarrow +\infty$.

Therefore, the sequence diverges to $+\infty$.

(8) $\frac{3^n - 1}{\sqrt{2^n}}$ [Sol] The general term is:

$$a_n = \frac{3^n - 1}{\sqrt{2^n}} = \left(\frac{3}{\sqrt{2}}\right)^n - \left(\frac{1}{\sqrt{2}}\right)^n.$$

When $n \rightarrow \infty$, $a_n \rightarrow +\infty$.

Therefore, the sequence diverges to $+\infty$.

N 96 a

Infinite Geometric Sequences 1

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	1	2	3-

Evaluate the following limits.

$$(1) \lim_{n \rightarrow \infty} \frac{1 + 3^{n+1}}{3^n + 2^n} = \lim_{n \rightarrow \infty} \frac{1 + 3^n \cdot 3}{3^n + 2^n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{3^n} + 3}{1 + \left(\frac{2}{3}\right)^n} = 3$$

$$(2) \lim_{n \rightarrow \infty} \frac{1 - 2^n}{2^{-n} + 2^n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{2^n} - 1}{\frac{1}{2^{2n}} + 1} = -1$$

$$(3) \lim_{n \rightarrow \infty} \frac{(\sqrt{3})^n}{3^n + 3^{2n}} = \lim_{n \rightarrow \infty} \frac{\left(\frac{\sqrt{3}}{9}\right)^n}{\left(\frac{1}{3}\right)^n + 1} = 0$$

$$(4) \lim_{n \rightarrow \infty} \frac{2^n - (\sqrt{5})^n}{(-3)^n - 2^{2n}} = \lim_{n \rightarrow \infty} \frac{\frac{1}{2^n} - \left(\frac{\sqrt{5}}{4}\right)^n}{\left(-\frac{3}{4}\right)^n - 1} = 0$$

$$(5) \lim_{n \rightarrow \infty} \frac{(\sqrt{6})^n + 3^{n+2}}{3^n + 3^{-n}} = \lim_{n \rightarrow \infty} \frac{(\sqrt{6})^n + 3^n \cdot 9}{3^n + \left(\frac{1}{3}\right)^n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{\sqrt{6}}{3}\right)^n + 9}{1 + \frac{1}{3^{2n}}} = 9$$

$$(6) \lim_{n \rightarrow \infty} \frac{1 \times 2 \times 2^2 \times \dots \times 2^n}{3^{n(n+1)}}$$

$$= \lim_{n \rightarrow \infty} \frac{2^{1+2+\dots+n}}{3^{n(n+1)}}$$

$$= \lim_{n \rightarrow \infty} \frac{2^{\frac{n(n+1)}{2}}}{3^{n(n+1)}}$$



$$3^{n(n+1)} = 3^{2 \left[\frac{n(n+1)}{2} \right]} = 9^{\frac{n(n+1)}{2}}$$

$$= \lim_{n \rightarrow \infty} \left(\frac{2}{9} \right)^{\frac{n(n+1)}{2}}$$

$$= 0$$

$$(7) \lim_{n \rightarrow \infty} \frac{4 \times 16 \times 64 \times \dots \times 4^n}{5^{n^2+n-2}}$$

$$= \lim_{n \rightarrow \infty} \frac{4^{1+2+3+\dots+n}}{5^{n^2+n-2}}$$

$$= \lim_{n \rightarrow \infty} \frac{4^{\frac{n(n+1)}{2}}}{5^{n^2+n-2}}$$

$$= \lim_{n \rightarrow \infty} \frac{2^{n(n+1)}}{5^{n^2+n-2}}$$

$$= 25 \cdot \lim_{n \rightarrow \infty} \frac{2^{n(n+1)}}{5^{n(n+1)}}$$

$$= 25 \cdot \lim_{n \rightarrow \infty} \left(\frac{2}{5} \right)^{n(n+1)}$$

$$= 0$$

Infinite Geometric Sequences 1

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	1	-	2-

1. Evaluate the limit $y = \lim_{n \rightarrow \infty} \frac{1}{2 + x^n}$, for each given case of x below.

If there is no limit value, write "none".

[Sol]

When $x > 1$, $y = 0$

When $x = 1$, $y = \frac{1}{3}$

When $-1 < x < 1$, $y = \frac{1}{2}$

When $x = -1$, $y = \text{none}$

When $x < -1$, $y = 0$

2. Evaluate the limit $y = \lim_{n \rightarrow \infty} \frac{x^n}{1 + x^n}$, ($x \neq -1$) for each different case of x .

If there is no limit value, write "none".

[Sol]

When $x > 1$, $y = 1$

When $x = 1$, $y = \frac{1}{2}$

When $-1 < x < 1$, $y = 0$

When $x = -1$, $y = \text{none}$

When $x < -1$, $y = 1$

Note Summary: In each of the above exercises, there are 3 different limit values.

These values can be found by analyzing the following cases:

- (a) When $|x| > 1$, (b) When $x = \text{span style="border: 1px solid black; padding: 0 5px;">1, (c) When $\text{span style="border: 1px solid black; padding: 0 5px;">x $< \text{span style="border: 1px solid black; padding: 0 5px;">1$$$

N 97 b

3. Evaluate the limit $y = \lim_{n \rightarrow \infty} \frac{2x^n}{x^n + 1}$ ($x \neq -1$).

[Sol]

(a) When $|x| > 1$, $y = 2$

(b) When $x = 1$, $y = 1$

(c) When $|x| < 1$, $y = 0$

4. Evaluate the limit $y = \lim_{n \rightarrow \infty} \frac{1}{x^n + 3}$ ($x \neq -1$).

[Sol]

(a) When $|x| > 1$, $y = 0$

(b) When $x = 1$, $y = \frac{1}{4}$

(c) When $|x| < 1$, $y = \frac{1}{3}$

5. Evaluate the limit $y = \lim_{n \rightarrow \infty} \frac{4x^n}{x^n + 1}$ ($x \neq -1$).

[Sol]

(a) When $|x| > 1$, $y = 4$

(b) When $x = 1$, $y = 2$

(c) When $|x| < 1$, $y = 0$

Time : to : Date Name

100%	90%	80%	70%	60%~
(mistakes) 0	-	-	-	1-

1. Evaluate the following limit.

$$\lim_{n \rightarrow \infty} \frac{a^{n+1} - 1}{a^n + 1} \quad (a \neq -1)$$

[Sol] Let $f(a) = \lim_{n \rightarrow \infty} \frac{a^{n+1} - 1}{a^n + 1}$.

(a) When $|a| < 1$, $\lim_{n \rightarrow \infty} a^{n+1} = \boxed{0}$, $\lim_{n \rightarrow \infty} a^n = \boxed{0}$ $\therefore f(a) = \boxed{-1}$

(b) When $|a| > 1$, $f(a) = \lim_{n \rightarrow \infty} \frac{a - \left(\frac{1}{a}\right)^n}{1 + \left(\frac{1}{a}\right)^n}$

Since $\left|\frac{1}{a}\right| < 1$, $\therefore f(a) = \boxed{a}$

(c) When $a = 1$, $\frac{a^{n+1} - 1}{a^n + 1} = 0$ $\therefore f(a) = \boxed{0}$

N 98 b

2. Evaluate the following limit.

$$\lim_{n \rightarrow \infty} \frac{2a^{n+2} + 1}{a^n + 1} \quad (a \neq -1)$$

[Sol] Let $f(a) = \lim_{n \rightarrow \infty} \frac{2a^{n+2} + 1}{a^n + 1}$.

(a) When $|a| < 1$, $\lim_{n \rightarrow \infty} a^{n+2} = 0$, $\lim_{n \rightarrow \infty} a^n = 0 \quad \therefore f(a) = 1$

(b) When $|a| > 1$, $f(a) = \lim_{n \rightarrow \infty} \frac{2a^2 + \left(\frac{1}{a}\right)^n}{1 + \left(\frac{1}{a}\right)^n}$

Since $\left|\frac{1}{a}\right| < 1$, $\therefore f(a) = 2a^2$

(c) When $a = 1$, $\frac{2a^{n+2} + 1}{a^n + 1} = \frac{3}{2} \quad \therefore f(a) = \frac{3}{2}$

Time : to : Date Name

100%	90%	80%	70%	60%~
(mistakes) 0	-	-	-	1-

Draw the graphs of the following functions.

$$(1) y = \lim_{n \rightarrow \infty} \frac{x^{n+1}}{1 + x^n}$$

[Sol]

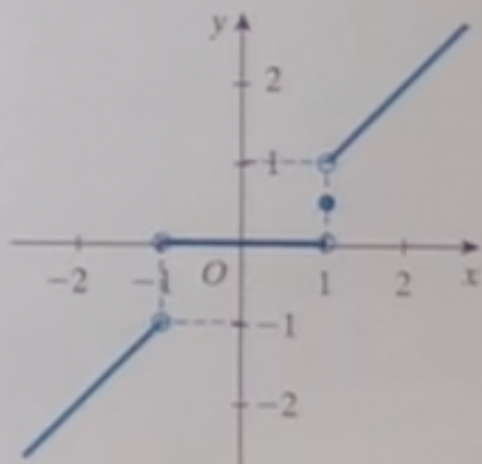
x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
y	-2	none	0	0	0	$\frac{1}{2}$	2

$$(a) \text{ When } |x| > 1, \quad y = \lim_{n \rightarrow \infty} \frac{x \cdot x^n}{1 + x^n} = \lim_{n \rightarrow \infty} \frac{x}{\left(\frac{1}{x}\right)^n + 1} = x$$

$$(b) \text{ When } |x| < 1, \quad y = \lim_{n \rightarrow \infty} \frac{x^{n+1}}{1 + x^n} = 0$$

$$(c) \text{ When } x = 1, \quad y = \lim_{n \rightarrow \infty} \frac{x^{n+1}}{1 + x^n} = \frac{1}{1 + 1} = \frac{1}{2}$$

$$(d) \text{ When } x = -1, \quad y \text{ has no value}$$



N 99 b

$$(2) \ y = \lim_{n \rightarrow \infty} \frac{x^n - 1}{x^n + 1}$$

[Sol]

(a) When $|x| > 1$,

$$y = \lim_{n \rightarrow \infty} \frac{x^n - 1}{x^n + 1} = \lim_{n \rightarrow \infty} \frac{1 - \left(\frac{1}{x}\right)^n}{1 + \left(\frac{1}{x}\right)^n} = 1$$

(b) When $|x| < 1$,

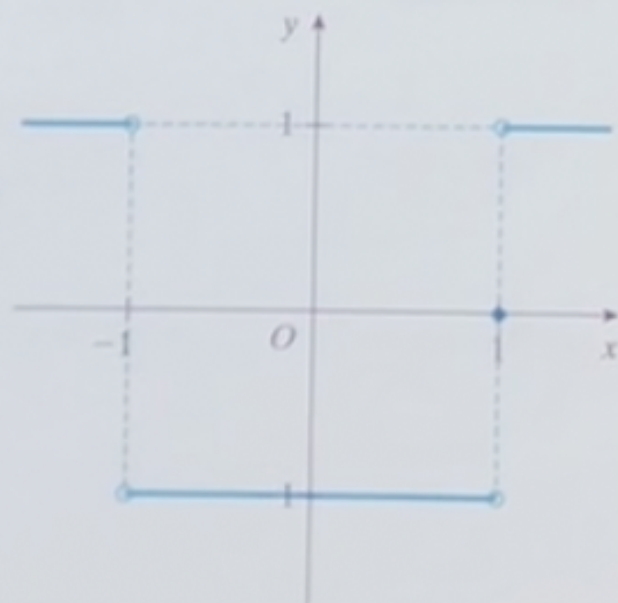
$$y = -1$$

(c) When $x = 1$,

$$y = 0$$

(d) When $x = -1$,

y has no value



Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	1	-	2-

1. Evaluate the following limits.

$$(1) \lim_{n \rightarrow \infty} \frac{5^n + 2^n}{3^{2n} + 4^n} = \lim_{n \rightarrow \infty} \frac{5^n + 2^n}{9^n + 4^n}$$

$$= \lim_{n \rightarrow \infty} \frac{\left(\frac{5}{9}\right)^n + \left(\frac{2}{9}\right)^n}{1 + \left(\frac{4}{9}\right)^n}$$

$$= 0$$

$$(2) \lim_{n \rightarrow \infty} \frac{6^{n+2} + 3^n}{2^{2n} + 6^n} = \lim_{n \rightarrow \infty} \frac{6^n \cdot 6^2 + 3^n}{4^n + 6^n}$$

$$= \lim_{n \rightarrow \infty} \frac{36 + \left(\frac{1}{2}\right)^n}{\left(\frac{2}{3}\right)^n + 1}$$

$$= 36$$

$$(3) \lim_{n \rightarrow \infty} \frac{(\sqrt{5})^n - 1}{2^n + 1} = \lim_{n \rightarrow \infty} \frac{\left(\frac{\sqrt{5}}{2}\right)^n - \left(\frac{1}{2}\right)^n}{1 + \left(\frac{1}{2}\right)^n}$$

$$= \infty$$

$$(4) \lim_{n \rightarrow \infty} \frac{1 + 2 + 2^2 + \dots + 2^{n-1}}{3^n} = \lim_{n \rightarrow \infty} \frac{2^n - 1}{3^n}$$

$$= \lim_{n \rightarrow \infty} \left[\left(\frac{2}{3}\right)^n - \left(\frac{1}{3}\right)^n \right]$$

$$= 0$$

2. Draw the graph of the following function.

$$f(x) = \lim_{n \rightarrow \infty} \frac{x^{2n+1}}{1 + x^{2n}}$$

[Sol]

(a) When $|x| > 1$,

$$f(x) = \lim_{n \rightarrow \infty} \frac{x^{2n+1}}{1 + x^{2n}} = \lim_{n \rightarrow \infty} \frac{x}{\left(\frac{1}{x}\right)^{2n} + 1} = x$$

(b) When $|x| < 1$,

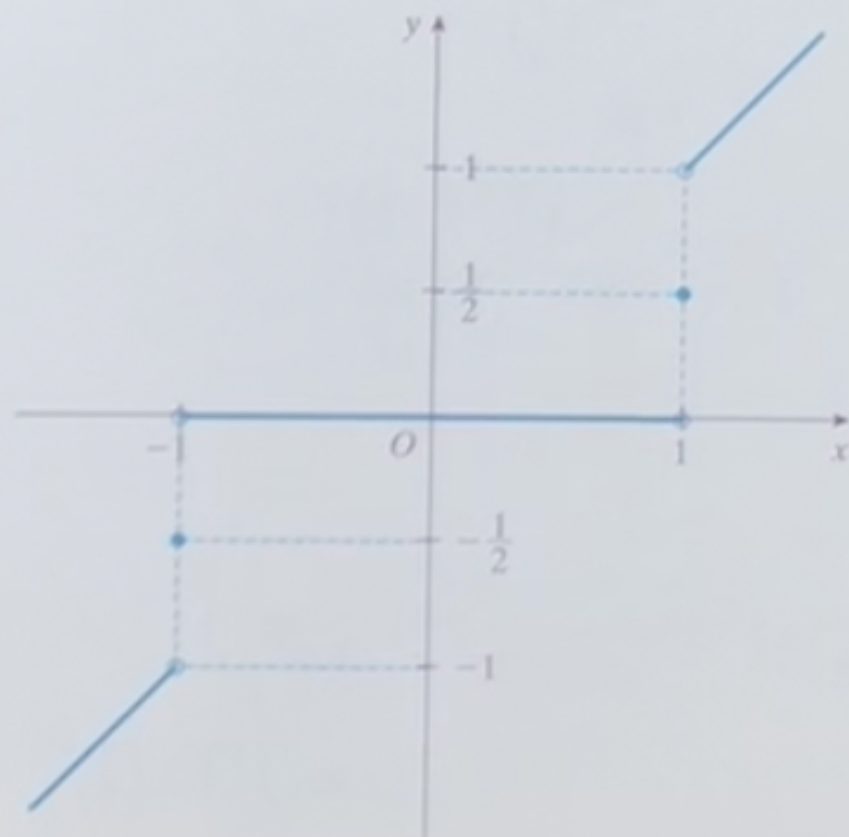
$$f(x) = \lim_{n \rightarrow \infty} \frac{x^{2n+1}}{1 + x^{2n}} = \frac{0}{1 + 0} = 0$$

(c) When $x = 1$,

$$f(x) = \lim_{n \rightarrow \infty} \frac{x^{2n+1}}{1 + x^{2n}} = \frac{1}{2}$$

(d) When $x = -1$,

$$f(x) = \lim_{n \rightarrow \infty} \frac{x^{2n+1}}{1 + x^{2n}} = \frac{-1}{1 + 1} = -\frac{1}{2}$$



Infinite Geometric Sequences 2

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	-	-	1-

1. Evaluate the following limit when $r > 0$.

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{r^n} - r^n}{r^n + \frac{1}{r^n}}$$

[Sol]

(a) When $r > 1$, $\lim_{n \rightarrow \infty} \frac{1}{r^n} = 0$

Therefore, $\lim_{n \rightarrow \infty} \frac{\frac{1}{r^n} - r^n}{r^n + \frac{1}{r^n}} = \lim_{n \rightarrow \infty} \frac{\frac{1}{r^{2n}} - 1}{1 + \frac{1}{r^{2n}}} = -1$

(b) When $r = 1$, $\lim_{n \rightarrow \infty} \frac{\frac{1}{r^n} - r^n}{r^n + \frac{1}{r^n}} = \frac{1 - 1}{1 + 1} = 0$

(c) When $0 < r < 1$, $\lim_{n \rightarrow \infty} r^n = 0$

Therefore, $\lim_{n \rightarrow \infty} \frac{\frac{1}{r^n} - r^n}{r^n + \frac{1}{r^n}} = \lim_{n \rightarrow \infty} \frac{1 - r^{2n}}{r^{2n} + 1} = 1$

N 101 b

2. Evaluate the following limit when $r > 0$.

$$\lim_{n \rightarrow \infty} \frac{r^n + 2}{r^n + \frac{1}{r^n}}$$

[Sol]

(a) When $r > 1$, $\lim_{n \rightarrow \infty} \frac{1}{r^n} = 0$

$$\text{Therefore, } \lim_{n \rightarrow \infty} \frac{r^n + 2}{r^n + \frac{1}{r^n}} = \lim_{n \rightarrow \infty} \frac{1 + 2 \cdot \frac{1}{r^n}}{1 + \frac{1}{r^{2n}}} = 1$$

(b) When $r = 1$, $\lim_{n \rightarrow \infty} \frac{r^n + 2}{r^n + \frac{1}{r^n}} = \frac{1 + 2}{1 + 1} = \frac{3}{2}$

(c) When $0 < r < 1$, $\lim_{n \rightarrow \infty} r^n = 0$

$$\text{Therefore, } \lim_{n \rightarrow \infty} \frac{r^n + 2}{r^n + \frac{1}{r^n}} = \lim_{n \rightarrow \infty} \frac{r^{2n} + 2r^n}{r^{2n} + 1} = 0$$

Time : to : Date Name

100%	90%	80%	70%	60%~
(mistakes) 0	-	-	-	-

1. Draw the graph of the function $y = f(x) = \lim_{n \rightarrow \infty} \frac{x^{2n-1} + x^2 + x}{x^{2n} + 1}$.

[Sol]

- (a) When $|x| < 1$, $\lim_{n \rightarrow \infty} x^{2n-1} = 0$, $\lim_{n \rightarrow \infty} x^{2n} = 0$

Therefore, $f(x) = \boxed{x^2 + x}$

- (b) When $|x| > 1$,

Dividing the numerator and the denominator by x^{2n} ,

$$f(x) = \lim_{n \rightarrow \infty} \frac{\frac{1}{x} + \frac{x^2 + x}{x^{2n}}}{\frac{x^{2n}}{x^{2n}} + \frac{1}{x^{2n}}} = \lim_{n \rightarrow \infty} \frac{\frac{1}{x} + (x^2 + x) \left(\frac{1}{x} \right)^{2n}}{1 + \left(\frac{1}{x} \right)^{2n}}$$

Since $\left| \frac{1}{x} \right| < 1$, $\lim_{n \rightarrow \infty} \left(\frac{1}{x} \right)^{2n} = 0$

Therefore, $f(x) = \boxed{\frac{1}{x}}$.

- (c) When $x = 1$,

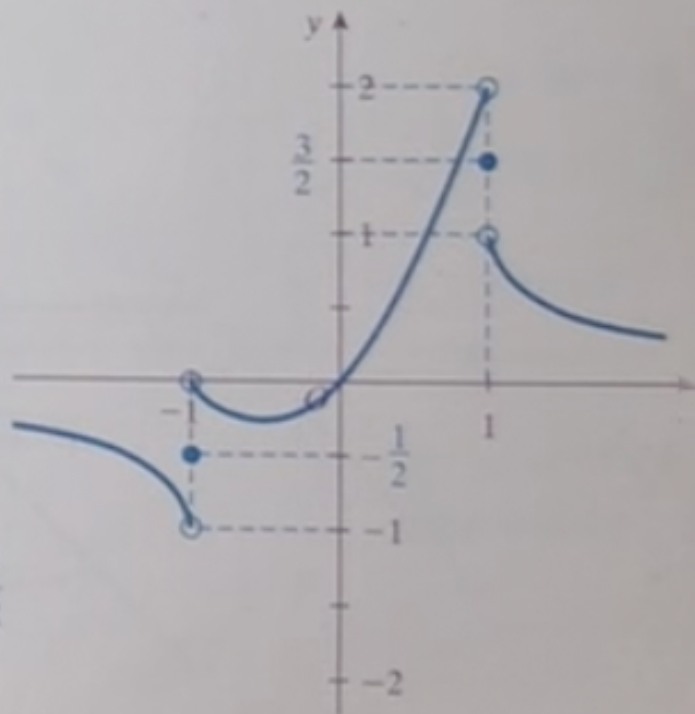
$$f(x) = \lim_{n \rightarrow \infty} \frac{x^{2n-1} + x^2 + x}{x^{2n} + 1} = \frac{3}{2}$$

Therefore, $f(1) = \frac{3}{2}$

- (d) When $x = -1$,

$$f(x) = \lim_{n \rightarrow \infty} \frac{x^{2n-1} + x^2 + x}{x^{2n} + 1} = -\frac{1}{2}$$

Therefore, $f(-1) = -\frac{1}{2}$



N 102 b

2. Draw the graph of the function $y = f(x) = \lim_{n \rightarrow \infty} \frac{x^{2n} + x + 1}{x^{2n-1} + 1}$.

[Sol]

- (a) When $|x| < 1$, $\lim_{n \rightarrow \infty} x^{2n-1} = 0$, $\lim_{n \rightarrow \infty} x^{2n} = 0$

Therefore, $f(x) = x + 1$

- (b) When $|x| > 1$,

Dividing the numerator and the denominator by x^{2n-1} ,

$$f(x) = \lim_{n \rightarrow \infty} \frac{\frac{x^{2n}}{x^{2n-1}} + \frac{x}{x^{2n-1}} + \frac{1}{x^{2n-1}}}{1 + \frac{1}{x^{2n-1}}} = \lim_{n \rightarrow \infty} \frac{x + x\left(\frac{1}{x}\right)^{2n-1} + \left(\frac{1}{x}\right)^{2n-1}}{1 + \left(\frac{1}{x}\right)^{2n-1}}$$

Since $\left|\frac{1}{x}\right| < 1$, $\lim_{n \rightarrow \infty} \left(\frac{1}{x}\right)^{2n-1} = 0$

Therefore, $f(x) = x$

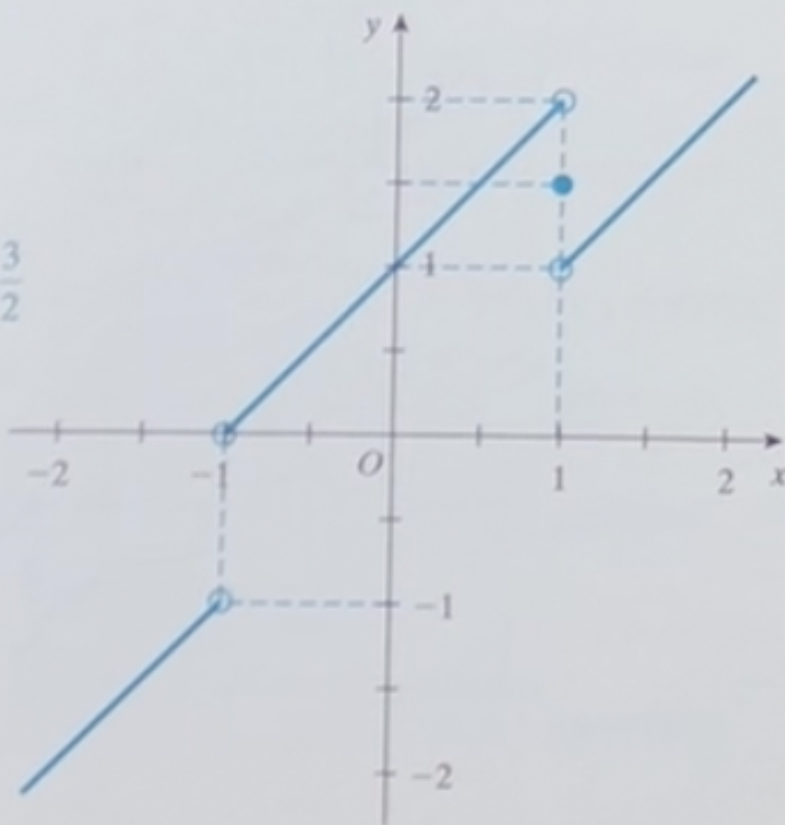
- (c) When $x = 1$,

$$f(x) = \lim_{n \rightarrow \infty} \frac{x^{2n} + x + 1}{x^{2n-1} + 1} = \frac{3}{2}$$

Therefore, $f(1) = \frac{3}{2}$

- (d) When $x = -1$,

y has no value



Infinite Geometric Sequences 2

Time : to : Date Name

100%	90%	80%	70%	60%~
(mistakes) 0	-	-	-	1-

1. Draw the graph of the function $y = \lim_{n \rightarrow \infty} \left(x \frac{x^{2n} - 1}{x^{2n} + 1} \right)$.

[Sol]

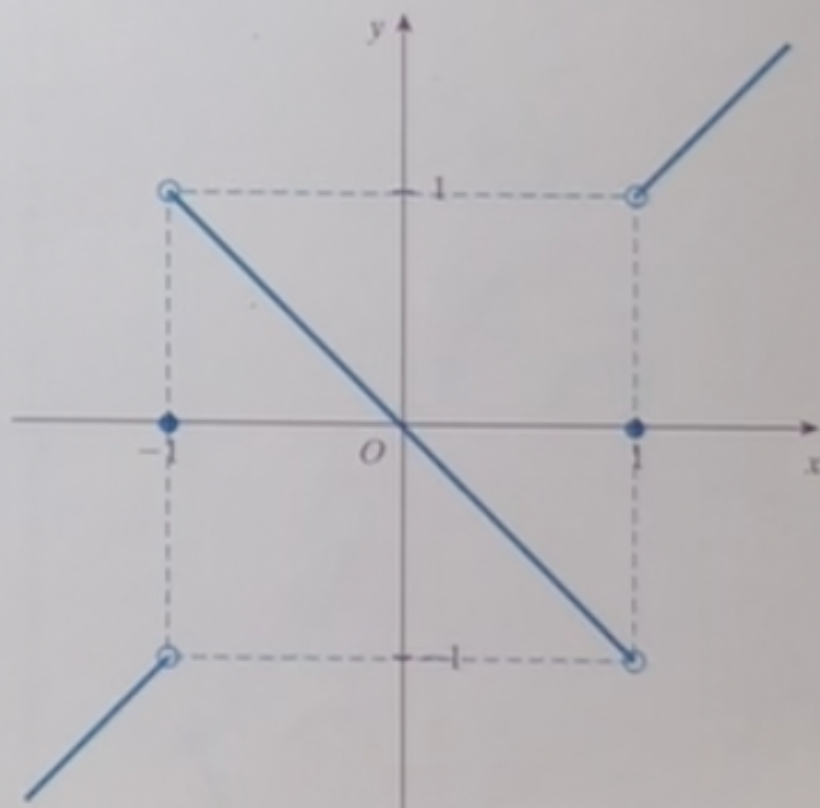
(a) When $|x| < 1$, $\lim_{n \rightarrow \infty} x^{2n} = 0$

Therefore, $y = \lim_{n \rightarrow \infty} \left(x \frac{x^{2n} - 1}{x^{2n} + 1} \right) = -x$

(b) When $|x| > 1$, $y = \lim_{n \rightarrow \infty} \left(x \frac{1 - \frac{1}{x^{2n}}}{1 + \frac{1}{x^{2n}}} \right) = x$

(c) When $x = 1$, $y = 0$

(d) When $x = -1$, $y = 0$



Infinite Geometric Sequences 2

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	—	—	—	—

1. Draw the graph of the function $y = \lim_{n \rightarrow \infty} \frac{x^n + 2x + 1}{x^{n-1} + 1}$.

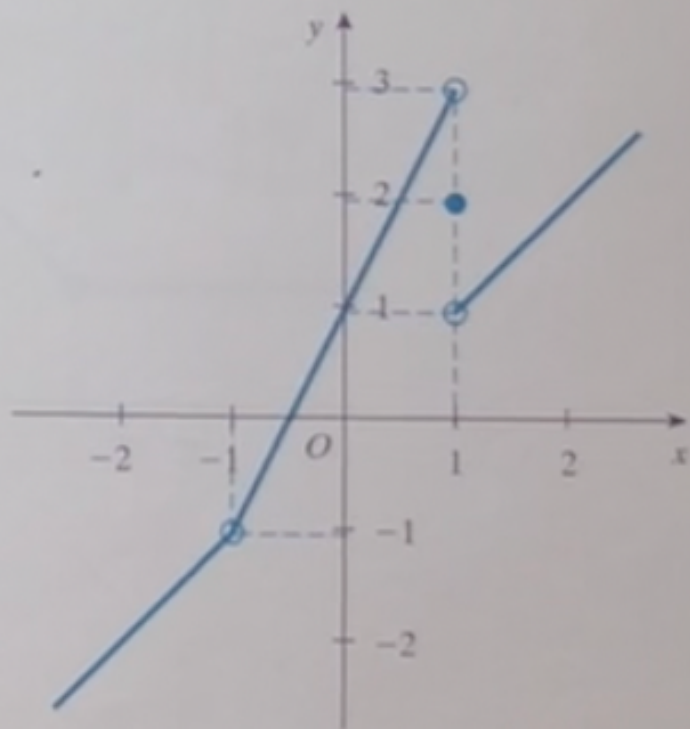
[Sol]

(a) When $|x| < 1$, $y = \lim_{n \rightarrow \infty} \frac{x^n + 2x + 1}{x^{n-1} + 1} = 2x + 1$

(b) When $|x| > 1$, $y = \lim_{n \rightarrow \infty} \frac{x + \frac{2}{x^{n-2}} + \frac{1}{x^{n-1}}}{1 + \frac{1}{x^{n-1}}} = x$

(c) When $x = 1$, $y = \frac{1 + 2 + 1}{1 + 1} = 2$

(d) When $x = -1$, y has no value



N 104 b

2. Draw the graph of the function $y = \lim_{n \rightarrow \infty} \frac{x^{n-1} + x + 2}{x^{n-1} + 1}$.

[Sol]

(a) When $|x| < 1$,

$$y = \lim_{n \rightarrow \infty} \frac{x^{n-1} + x + 2}{x^{n-1} + 1} = x + 2$$

(b) When $|x| > 1$,

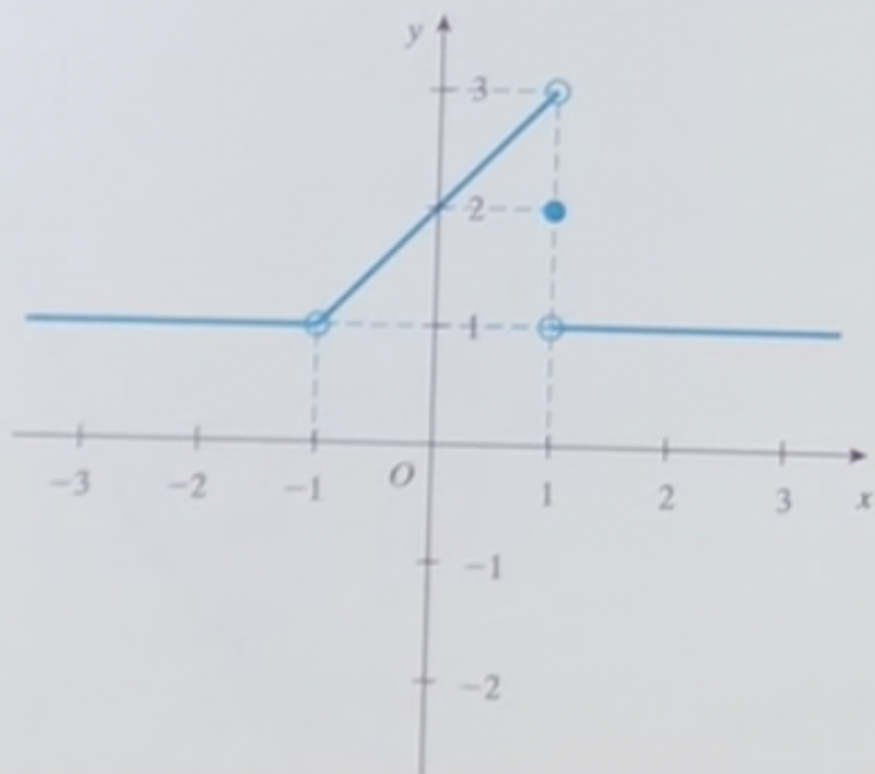
$$y = \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{x^{n-2}} + \frac{2}{x^{n-1}}}{1 + \frac{1}{x^{n-1}}} = 1$$

(c) When $x = 1$,

$$y = \frac{1 + 1 + 2}{1 + 1} = 2$$

(d) When $x = -1$,

y has no value



Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	1	-	2

Evaluate the following limits.

$$(1) \lim_{n \rightarrow \infty} \log_{10} \frac{n+1}{n} = \lim_{n \rightarrow \infty} \log_{10} \left(1 + \frac{1}{n} \right) = 0$$

$$(2) \lim_{n \rightarrow \infty} [\log_{10}(2n+1) - \log_{10} n] = \lim_{n \rightarrow \infty} \log_{10} \left(2 + \frac{1}{n} \right) \\ = \log_{10} 2$$

$$(3) \lim_{n \rightarrow \infty} \frac{\log_{10}(2^n + 3^n)}{n} = \lim_{n \rightarrow \infty} \left\{ \frac{1}{n} \log_{10} \left(3^n \left[\left(\frac{2}{3} \right)^n + 1 \right] \right) \right\} \\ = \lim_{n \rightarrow \infty} \left\{ \log_{10} 3 + \frac{1}{n} \log_{10} \left[\left(\frac{2}{3} \right)^n + 1 \right] \right\} \\ = \log_{10} 3$$

$$(4) \lim_{n \rightarrow \infty} \log_2 \sqrt{\frac{8n+3}{n}} = \lim_{n \rightarrow \infty} \log_2 \sqrt{8 + \frac{3}{n}} \\ = \log_2 \sqrt{8} \\ = \frac{3}{2}$$

N 105 b

$$(5) \lim_{n \rightarrow \infty} \left[\frac{1}{n} \log_{10}(a^n + a^{2n}) \right] \quad (a > 0)$$

$$\begin{aligned} \text{[Sol]} \quad \lim_{n \rightarrow \infty} \left[\frac{1}{n} \log_{10}(a^n + a^{2n}) \right] &= \lim_{n \rightarrow \infty} \left\{ \frac{1}{n} \log_{10}[a^n(1 + a^n)] \right\} \\ &= \lim_{n \rightarrow \infty} (\log_{10} a + \log_{10} \sqrt[n]{1 + a^n}) \end{aligned}$$

(a) When $0 < a \leq 1$, $0 < a^n \leq 1$, and $1 < \sqrt[n]{1 + a^n} \leq \sqrt[n]{2}$

$$\lim_{n \rightarrow \infty} \sqrt[n]{2} = \lim_{n \rightarrow \infty} 2^{\frac{1}{n}} = 2^0 = 1$$

Therefore, $\lim_{n \rightarrow \infty} \sqrt[n]{1 + a^n} = 1$

$$\therefore \lim_{n \rightarrow \infty} \left[\frac{1}{n} \log_{10}(a^n + a^{2n}) \right] = \log_{10} a$$

(b) When $a > 1$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left[\frac{1}{n} \log_{10}(a^n + a^{2n}) \right] &= \lim_{n \rightarrow \infty} \left\{ \frac{1}{n} \log_{10} \left[a^{2n} \left(\frac{1}{a^n} + 1 \right) \right] \right\} \\ &= \lim_{n \rightarrow \infty} \left(\log_{10} a^2 + \log_{10} \sqrt[n]{\frac{1}{a^n} + 1} \right) \end{aligned}$$

Since $1 < \sqrt[n]{\frac{1}{a^n} + 1} < \sqrt[n]{2}$,

$$\lim_{n \rightarrow \infty} \sqrt[n]{2} = 1$$

$$\therefore \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{a^n} + 1} = 1$$

$$\begin{aligned} \text{Therefore, } \lim_{n \rightarrow \infty} \left\{ \frac{1}{n} \log_{10} \left[a^{2n} \left(\frac{1}{a^n} + 1 \right) \right] \right\} &= \log_{10} a^2 + \log_{10} 1 \\ &= 2 \log_{10} a \end{aligned}$$

Infinite Geometric Sequences 2

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	1	2	3	4

In each of the following exercises, derive the first 6 terms of the sequence with the given general term, and then state whether the sequence converges, diverges, or oscillates.

Complete and check your answers.

$$\sin \frac{n\pi}{2}$$

[Sol] $\sin \frac{\pi}{2}, \sin \pi, \sin \frac{3\pi}{2}, \sin 2\pi, \sin \frac{5\pi}{2}, \sin 3\pi, \dots$

$1, 0, -1, 0, 1, 0, \dots$

The sequence **oscillates**.

Answer: $\frac{3\pi}{2}, 2\pi, \frac{5\pi}{2}, 3\pi, 0, 1, 0, \text{oscillates}$

(1) $\cos \frac{n\pi}{2}$

[Sol] $\cos \frac{\pi}{2}, \cos \pi, \cos \frac{3\pi}{2}, \cos 2\pi, \cos \frac{5\pi}{2}, \cos 3\pi, \dots$

$0, -1, 0, 1, 0, -1, \dots$

The sequence oscillates.

(2) $\sin \frac{n\pi}{4}$

[Sol] $\sin \frac{\pi}{4}, \sin \frac{\pi}{2}, \sin \frac{3\pi}{4}, \sin \pi, \sin \frac{5\pi}{4}, \sin \frac{3\pi}{2}, \dots$

$\frac{\sqrt{2}}{2}, 1, \frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}, -1, \dots$

The sequence oscillates.

N 106 b

(3) $\cos \frac{n\pi}{4}$

[Sol] $\cos \frac{\pi}{4}, \cos \frac{\pi}{2}, \cos \frac{3\pi}{4}, \cos \pi, \cos \frac{5\pi}{4}, \cos \frac{3\pi}{2}, \dots$

$$\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}, -1, -\frac{\sqrt{2}}{2}, 0, \dots$$

The sequence oscillates.

(4) $\cos n\pi$

[Sol] $\cos \pi, \cos 2\pi, \cos 3\pi, \cos 4\pi, \cos 5\pi, \cos 6\pi, \dots$

$$-1, 1, -1, 1, -1, 1, \dots$$

The sequence oscillates.

(5) $(-1)^n \sin \frac{n\pi}{2}$

[Sol] $-\sin \frac{\pi}{2}, \sin \pi, -\sin \frac{3\pi}{2}, \sin 2\pi, -\sin \frac{5\pi}{2}, \sin 3\pi, \dots$

$$-1, 0, 1, 0, -1, 0, \dots$$

The sequence oscillates.

(6) $n + |\cos n\pi|$

[Sol] $2, 3, 4, 5, 6, 7, \dots$

The sequence diverges (to positive infinity).

(7) $-n + \sin n\pi$

[Sol] $-1, -2, -3, -4, -5, -6, \dots$

The sequence diverges (to negative infinity).

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	1	2	3-

In each of the following exercises, indicate whether the sequence of the given general term converges, diverges, or oscillates. If it converges, state the limit value.

(1) $\sin \frac{n\pi}{3}$

[Sol] $\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0, \dots$

The sequence oscillates.

(2) $\sin \frac{\pi}{n}$

[Sol] As $n \rightarrow \infty$, $\frac{\pi}{n} \rightarrow 0$

Therefore, $\sin \frac{\pi}{n} \rightarrow 0$

And, thus, the sequence converges to limit value 0.

(3) $\cos \frac{\pi}{n}$

[Sol] As $n \rightarrow \infty$, $\frac{\pi}{n} \rightarrow 0$

Therefore, $\cos \frac{\pi}{n} \rightarrow 1$

And, thus, the sequence converges to limit value 1.

(4) $\cos \frac{2\pi}{n}$

[Sol] As $n \rightarrow \infty$, $\frac{2\pi}{n} \rightarrow 0$

Therefore, $\cos \frac{2\pi}{n} \rightarrow 1$

And, thus, the sequence converges to limit value 1.

N 107 b

(5) $(-1)^n \cos \frac{n\pi}{2}$

[Sol] $0, -1, 0, 1, 0, -1, 0, 1, \dots$

The sequence oscillates.

(6) $\sin\left(n\pi + \frac{\pi}{4}\right)$

[Sol] $-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \dots$

The sequence oscillates.

(7) $n \cos \frac{\pi}{n}$

[Sol] As $n \rightarrow \infty$, $\frac{\pi}{n} \rightarrow 0$

Therefore, $\cos \frac{\pi}{n} \rightarrow 1$

And, thus, the sequence diverges (to positive infinity).

(8) $\frac{\cos n}{n}$

[Sol] As $n \rightarrow \infty$, $\frac{\cos n}{n} \rightarrow 0$

And, thus, the sequence converges to limit value 0.

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	1	2	3	4

1. In each of the following exercises, indicate whether the sequence of the given general term converges, diverges, or oscillates. If it converges, state the limit value.

(1) $\sin \frac{\pi}{2n}$

[Sol] As $n \rightarrow \infty$, $\frac{\pi}{2n} \rightarrow 0$

$$\sin \frac{\pi}{2n} \rightarrow 0$$

Therefore, the sequence converges to 0.

(2) $\cos \frac{\pi}{2n}$

[Sol] As $n \rightarrow \infty$, $\frac{\pi}{2n} \rightarrow 0$

$$\cos \frac{\pi}{2n} \rightarrow 1$$

Therefore, the sequence converges to 1.

(3) $\sin \left(\frac{\pi}{n} + \pi \right)$

[Sol] As $n \rightarrow \infty$, $\left(\frac{\pi}{n} + \pi \right) \rightarrow \pi$

$$\sin \left(\frac{\pi}{n} + \pi \right) \rightarrow 0$$

Therefore, the sequence converges to 0.

(4) $\cos \left(\frac{\pi}{n} + \pi \right)$

[Sol] As $n \rightarrow \infty$, $\left(\frac{\pi}{n} + \pi \right) \rightarrow \pi$

$$\cos \left(\frac{\pi}{n} + \pi \right) \rightarrow -1$$

Therefore, the sequence converges to -1.

N 108 b

2. In each of the following exercises, given that $0 < \theta < \frac{\pi}{2}$, indicate whether the sequence of the given general term converges, diverges, or oscillates. If it converges, state the limit value.

(1) $\sin \frac{\theta}{n}$

[Sol] As $n \rightarrow \infty$, $\frac{\theta}{n} \rightarrow 0$

$$\sin \frac{\theta}{n} \rightarrow 0$$

Therefore, the sequence converges to 0.

(2) $\sin^n \theta$

[Sol] As $n \rightarrow \infty$,

$$\sin^n \theta = (\sin \theta)^n \rightarrow 0 \quad (\because 0 < \sin \theta < 1)$$

Therefore, the sequence converges to 0.

(3) $\cos^n \theta$

[Sol] As $n \rightarrow \infty$,

$$\cos^n \theta = (\cos \theta)^n \rightarrow 0 \quad (\because 0 < \cos \theta < 1)$$

Therefore, the sequence converges to 0.

(4) $\sin(n\theta)$

[Sol] As $n \rightarrow \infty$,

$\sin(n\theta)$ does not converge to a limit.

Therefore, the sequence oscillates.

Infinite Geometric Sequences 2

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	—	—	1	2

For any value of n , when $a_n < b_n$, $\lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} b_n$

Given the sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$, where $a_n \leq c_n \leq b_n$ ($n = 1, 2, 3, \dots$), if $\{a_n\}$ and $\{b_n\}$ both converge to L , then the sequence $\{c_n\}$ also converges to L . $\therefore \lim_{n \rightarrow \infty} c_n = L$

1. Evaluate the following limits.

(1) $\lim_{n \rightarrow \infty} \frac{\sin n\theta}{n}$

[Sol] Since $-1 \leq \sin n\theta \leq 1$,

$$-\frac{1}{n} \leq \frac{\sin n\theta}{n} \leq \frac{1}{n}$$

$$\text{Therefore, } \lim_{n \rightarrow \infty} \left(-\frac{1}{n} \right) \leq \lim_{n \rightarrow \infty} \frac{\sin n\theta}{n} \leq \lim_{n \rightarrow \infty} \frac{1}{n}$$

$$\lim_{n \rightarrow \infty} \left(-\frac{1}{n} \right) = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

$$\therefore \lim_{n \rightarrow \infty} \frac{\sin n\theta}{n} = 0$$

(2) $\lim_{n \rightarrow \infty} \frac{\cos n\pi}{n+1}$

[Sol] Since $-1 \leq \cos n\pi \leq 1$,

$$-\frac{1}{n+1} \leq \frac{\cos n\pi}{n+1} \leq \frac{1}{n+1}$$

$$\text{Therefore, } \lim_{n \rightarrow \infty} \left(\frac{-1}{n+1} \right) \leq \lim_{n \rightarrow \infty} \frac{\cos n\pi}{n+1} \leq \lim_{n \rightarrow \infty} \frac{1}{n+1}$$

$$\lim_{n \rightarrow \infty} \left(\frac{-1}{n+1} \right) = \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0$$

$$\therefore \lim_{n \rightarrow \infty} \frac{\cos n\pi}{n+1} = 0$$

N 109 b

2. Given that $0 < \theta < \pi$ where $\theta \neq \frac{\pi}{2}$, evaluate the following limit.

$$\lim_{n \rightarrow \infty} \frac{\cos^n \theta + 1}{\sin^n \theta + 1}$$

[Sol] $0 < \sin \theta < 1$, $0 < |\cos \theta| < 1$,

When $n \rightarrow \infty$, $\sin^n \theta \rightarrow \boxed{0}$

When $n \rightarrow \infty$, $\cos^n \theta \rightarrow \boxed{0}$

$$\therefore \lim_{n \rightarrow \infty} \frac{\cos^n \theta + 1}{\sin^n \theta + 1} = 1$$

3. Given that $\frac{\pi}{2} < \theta < \pi$ where $\theta \neq \frac{3}{4}\pi$, evaluate the following limit.

$$\lim_{n \rightarrow \infty} \frac{\sin^n 2\theta}{\cos^n \theta - 3}$$

[Sol] $0 < |\sin 2\theta| < 1$, $0 < |\cos \theta| < 1$

When $n \rightarrow \infty$, $\sin^n 2\theta \rightarrow 0$

When $n \rightarrow \infty$, $\cos^n \theta \rightarrow 0$

$$\therefore \lim_{n \rightarrow \infty} \frac{\sin^n 2\theta}{\cos^n \theta - 3} = 0$$

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	—	—	1	2

1. Evaluate the following limit when $a > 0$.

$$\lim_{n \rightarrow \infty} \frac{a^{-n} - a^n}{a^n + a^{-n}}$$

[Sol]

(a) When $a > 1$, $\lim_{n \rightarrow \infty} a^{-n} = \lim_{n \rightarrow \infty} \frac{1}{a^n} = 0$

$$\lim_{n \rightarrow \infty} \frac{a^{-n} - a^n}{a^n + a^{-n}} = \lim_{n \rightarrow \infty} \frac{a^{-2n} - 1}{1 + a^{-2n}} = -1 \quad (\because \lim_{n \rightarrow \infty} a^{-2n} = 0)$$

(b) When $a = 1$, $\lim_{n \rightarrow \infty} \frac{a^{-n} - a^n}{a^n + a^{-n}} = 0$

(c) When $0 < a < 1$, $\lim_{n \rightarrow \infty} \frac{a^{-n} - a^n}{a^n + a^{-n}} = \lim_{n \rightarrow \infty} \frac{1 - a^{2n}}{a^{2n} + 1} = 1 \quad (\because \lim_{n \rightarrow \infty} a^{2n} = 0)$

N 110 b

2. In each of the following exercises, indicate whether the sequence of the given general term converges, diverges, or oscillates. If it converges, state the limit value.

(1) $\cos \frac{n\pi}{3}$

[Sol] $\frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1, \dots$

The sequence oscillates.

(2) $\sin \frac{2\pi}{n}$

[Sol] As $n \rightarrow \infty$, $\frac{2\pi}{n} \rightarrow 0$

Therefore, $\sin \frac{2\pi}{n} \rightarrow 0$

And, thus, the sequence converges to limit value 0.

3. Evaluate the following limit.

$$\lim_{n \rightarrow \infty} \left(\frac{1}{n} \sin \frac{n\pi}{2} \right)$$

[Sol] Since $-1 \leq \sin \frac{n\pi}{2} \leq 1$,

$$-\frac{1}{n} \leq \frac{1}{n} \sin \frac{n\pi}{2} \leq \frac{1}{n}$$

Therefore, $\lim_{n \rightarrow \infty} \left(-\frac{1}{n} \right) \leq \lim_{n \rightarrow \infty} \left(\frac{1}{n} \sin \frac{n\pi}{2} \right) \leq \lim_{n \rightarrow \infty} \frac{1}{n}$

$$\lim_{n \rightarrow \infty} \left(-\frac{1}{n} \right) = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

$$\therefore \lim_{n \rightarrow \infty} \left(\frac{1}{n} \sin \frac{n\pi}{2} \right) = 0$$

Infinite Geometric Series

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	-	-	1-

Given the infinite geometric sequence $1, \frac{2}{3}, \left(\frac{2}{3}\right)^2, \dots, \left(\frac{2}{3}\right)^{n-1}, \dots$

The first term is $\boxed{1}$, and the common ratio is $\boxed{\frac{2}{3}}$.

Letting S_n denote the sum of the first n terms,

$$S_n = 1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \dots + \left(\frac{2}{3}\right)^{n-1}$$

$$S_n = \frac{1 - \left(\frac{2}{3}\right)^n}{1 - \frac{2}{3}} = 3 \left[1 - \boxed{\left(\frac{2}{3}\right)^n} \right]$$

Therefore, as $n \rightarrow \infty$, $S_n \rightarrow \boxed{3}$.

Answer: $1, \frac{2}{3}, \left(\frac{2}{3}\right)^2, \dots$

Given an infinite geometric sequence:

$$a, ar, ar^2, \dots, ar^{n-1}, \dots \text{ (where } a \neq 0 \text{ and } r \neq 0 \text{)}$$

The sum of all the terms is called an *Infinite Geometric Series*:

$$a + ar + ar^2 + \dots + ar^{n-1} + \dots \quad \dots \textcircled{1}$$

The sum of the first n terms is called a *Partial Sum*:

$$S_n = a + ar + \dots + ar^{n-1}$$

Given the infinite sequence $\{S_n\}$ formed by successive partial terms:

$$a, a + ar, a + ar + ar^2, \dots$$

- When the sequence converges, series $\textcircled{1}$ is said to **converge**.
- When the sequence diverges, series $\textcircled{1}$ is said to **diverge**.

N 111 b

In each of the following exercises, determine whether the given infinite geometric series converges or diverges. If it converges, find the sum of the terms.

Ex. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$

[Sol] Letting S_n be the partial sum of the first n terms,

The first term is $\boxed{1}$, and the common ratio is $\boxed{\frac{1}{2}}$.

$$S_n = 2 \left[1 - \left(\frac{1}{2} \right)^n \right]$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \boxed{2} \text{ (converges)}$$

Answers: 1, $\frac{1}{2}$, $\left(\frac{1}{2}\right)^n$, 2

(1) $1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \dots$

$$[\text{Sol}] S_n = \frac{\left(\frac{3}{2}\right)^n - 1}{\frac{3}{2} - 1} = 2 \left[\left(\frac{3}{2}\right)^n - 1 \right]$$

$$\therefore \lim_{n \rightarrow \infty} S_n = +\infty \text{ (diverges)}$$

(3) $1 - \frac{3}{2} + \frac{9}{4} - \frac{27}{8} + \dots$

$$[\text{Sol}] S_n = \frac{1 - \left(-\frac{3}{2}\right)^n}{1 + \frac{3}{2}} = \frac{2}{5} \left[1 - \left(-\frac{3}{2}\right)^n \right]$$

$$\therefore \lim_{n \rightarrow \infty} S_n \text{ oscillates (diverges)}$$

(2) $1 - \frac{3}{4} + \frac{9}{16} - \frac{27}{64} + \dots$

$$[\text{Sol}] S_n = \frac{1 - \left(-\frac{3}{4}\right)^n}{1 + \frac{3}{4}} = \frac{4}{7} \left[1 - \left(-\frac{3}{4}\right)^n \right]$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \frac{4}{7} \text{ (converges)}$$

Infinite Geometric Series

Time : to : Date Name

100%	90%	80%	70%	69%
(mistakes) 0	-	1	-	2-

Given $a + ar + ar^2 + \dots + ar^{n-1} + \dots$,Letting S_n be the partial sum of the first n terms,

$$S_n = \frac{a(1-r^n)}{1-r} \quad \text{where } r \neq 1$$

- (a) When $|r| < 1$, $\text{as } n \rightarrow \infty, r^n \rightarrow 0$ $\therefore \lim_{n \rightarrow \infty} S_n = \frac{a}{1-r}$
- (b) When $r \leq -1$ or $r > 1$, $\{r^n\}$ diverges $\therefore S_n$ diverges
- (c) When $r = 1$, $\text{as } n \rightarrow \infty, S_n = na$ $\therefore S_n$ diverges

Sum of an
Infinite
Geometric
Series

The infinite geometric series

$$\sum_{n=1}^{\infty} (ar^{n-1}) = a + ar + ar^2 + \dots + ar^{n-1} + \dots \quad (a \neq 0)$$

converges only when $|r| < 1$, and the sum of the terms, S , is expressed as:

$$S = \frac{a}{1-r}$$

In each of the following exercises, determine whether the given series converges or diverges. If it converges, find the sum.

(1) $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \dots$

[Sol] Since the common ratio $r = -\frac{1}{3}$ and $|r| < 1$,

this series **converges**.

$$S = \frac{1}{1 - \left(-\frac{1}{3}\right)} = \frac{3}{4}$$

N 112 b

$$(2) \quad 0.5 - 0.05 + 0.005 - 0.0005 + \dots$$

[Sol] Since the common ratio $r = -0.1$ and $|r| < 1$, this series converges.

$$S = \frac{0.5}{1 + 0.1} = \frac{5}{11}$$

$$(3) \quad -1 - 2 - 4 - 8 - \dots$$

[Sol] The common ratio $r = 2$ and $|r| > 1$.

Therefore, this series diverges.

$$(4) \quad 2 + 1 + \frac{1}{2} + \dots$$

[Sol] Since the common ratio $r = \frac{1}{2}$ and $|r| < 1$, this series converges.

$$S = \frac{2}{1 - \frac{1}{2}} = 4$$

$$(5) \quad 6 + (-2\sqrt{3}) + 2 + \dots$$

[Sol] Since the common ratio $r = \frac{-2\sqrt{3}}{6} = -\frac{\sqrt{3}}{3}$ and $|r| < 1$, this series converges.

$$S = \frac{6}{1 + \frac{\sqrt{3}}{3}} = \frac{18(3 - \sqrt{3})}{(3 + \sqrt{3})(3 - \sqrt{3})} = 3(3 - \sqrt{3})$$

$$(6) \quad (\sqrt{2} + 1) + (\sqrt{2} - 1) + (5\sqrt{2} - 7) + \dots$$

[Sol] Since the common ratio $r = \frac{\sqrt{2} - 1}{\sqrt{2} + 1} = 3 - 2\sqrt{2}$ and $|r| < 1$, this series converges.

$$S = \frac{\sqrt{2} + 1}{1 - (3 - 2\sqrt{2})} = \frac{(\sqrt{2} + 1)^2}{-2(1 - \sqrt{2})(1 + \sqrt{2})} = \frac{3 + 2\sqrt{2}}{2}$$

Infinite Geometric Series

Time : to : Date Name

100%	90%	80%	70%	60%~
(mistakes) 0	-	1	-	2-

Determine the sum, S , of each of the following infinite geometric series.

Ex. $\sum_{n=1}^{\infty} \frac{2}{3^{n-1}} \quad \left(= 2 + \frac{2}{3} + \frac{2}{9} + \dots \right)$

[Sol] $S = \frac{2}{1 - \frac{1}{3}} = 3$
 The first term is 2.
 The common ratio is $\frac{1}{3}$.

(1) $\sum_{n=1}^{\infty} \left[2 \left(\frac{4}{5} \right)^{n-1} \right]$

[Sol] $S = \frac{2}{1 - \frac{4}{5}} = 10$

(2) $\sum_{n=1}^{\infty} \left[(-1)^n \left(\frac{3}{4} \right)^{n+1} \right] = \frac{3}{4} \sum_{n=1}^{\infty} \left(-\frac{3}{4} \right)^n$

[Sol] $S = \frac{3}{4} \cdot \frac{-\frac{3}{4}}{1 + \frac{3}{4}} = -\frac{9}{28}$

(3) $9 + 3\sqrt{3} + 3 + \dots$

[Sol] The first term is $\boxed{9}$. The common ratio is $\boxed{\frac{\sqrt{3}}{3}}$.

$$S = \frac{9}{1 - \frac{1}{\sqrt{3}}} = \frac{9}{2}(3 + \sqrt{3})$$

(4) $(\sqrt{3} - 1) + (2 - \sqrt{3}) + \frac{3\sqrt{3} - 5}{2} + \dots$

[Sol] The first term is $\sqrt{3} - 1$.

The common ratio is $\frac{2 - \sqrt{3}}{\sqrt{3} - 1} = \frac{(2 - \sqrt{3})(\sqrt{3} + 1)}{(\sqrt{3} - 1)(\sqrt{3} + 1)} = \frac{-1 + \sqrt{3}}{2}$

$$S = \frac{\sqrt{3} - 1}{1 - \frac{-1 + \sqrt{3}}{2}} = \frac{2\sqrt{3}}{3}$$

N 113 b

$$(5) \ x + x^2 + x^3 + \dots$$

(where $|x| < 1$)

[Sol] (a) When $x \neq 0$,

The first term is x . The common ratio is x .

$$S = \frac{x}{1-x}$$

(b) When $x = 0$, each term becomes 0.

Therefore, $S = 0$

$$(6) \ x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \dots$$

[Sol] (a) When $x \neq 0$, $\left| \frac{1}{1+x^2} \right| < 1$

Therefore, the series converges, and the sum is:

$$S = \frac{x^2}{1 - \frac{1}{1+x^2}} = 1 + x^2$$

(b) When $x = 0$, each term becomes 0.

Therefore, $S = 0$

Infinite Geometric Series

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	—	—	—	—

Evaluate the following sums.

$$(1) \sum_{n=1}^{\infty} \left[\left(\frac{1}{2} \right)^n \sin \frac{n\pi}{2} \right]$$

[Sol] Since $\sin \frac{\pi}{2} = 1$, $\sin \pi = 0$, $\sin \frac{3\pi}{2} = -1$, $\sin 2\pi = 0$,
 $\sin \frac{5\pi}{2} = 1$, $\sin 3\pi = 0$, $\sin \frac{7\pi}{2} = -1$, $\sin 4\pi = 0$,

Letting m be a variable, where $m = 0, 1, 2, \dots$,

$$\sin \frac{n\pi}{2} = \begin{cases} 1 & \dots & n = 4m + 1 \\ 0 & \dots & n = 4m + 2 \\ -1 & \dots & n = 4m + 3 \\ 0 & \dots & n = 4m + 4 \end{cases}$$

Therefore, $\sum_{n=1}^{\infty} \left[\left(\frac{1}{2} \right)^n \sin \frac{n\pi}{2} \right] = \frac{1}{2} - \left(\frac{1}{2} \right)^3 + \left(\frac{1}{2} \right)^5 - \left(\frac{1}{2} \right)^7 + \dots$

is an infinite geometric series with first term $\frac{1}{2}$ and

common ratio $-\left(\frac{1}{2} \right)^2$.

$$\therefore \sum_{n=1}^{\infty} \left[\left(\frac{1}{2} \right)^n \sin \frac{n\pi}{2} \right] = \frac{\frac{1}{2}}{1 + \frac{1}{4}} = \frac{2}{5}$$

N 114 b

$$(2) \sum_{n=1}^{\infty} \left[\left(\frac{1}{3} \right)^n \cos \frac{n\pi}{2} \right]$$

$$[\text{Sol}] \cos \frac{n\pi}{2} = \begin{cases} 0 & \dots & n = 4m + 1 \\ -1 & \dots & n = 4m + 2 \\ 0 & \dots & n = 4m + 3 \\ 1 & \dots & n = 4m + 4 \end{cases}$$

$$(m = 0, 1, 2, \dots)$$

$$\text{Therefore, } \sum_{n=1}^{\infty} \left[\left(\frac{1}{3} \right)^n \cos \frac{n\pi}{2} \right] = 0 - \frac{1}{3^2} + 0 + \frac{1}{3^4} + 0 - \frac{1}{3^6} + \dots$$

is an infinite geometric series with first term $-\frac{1}{9}$ and

common ratio $-\frac{1}{9}$.

$$\therefore \sum_{n=1}^{\infty} \left[\left(\frac{1}{3} \right)^n \cos \frac{n\pi}{2} \right] = \frac{-\frac{1}{9}}{1 + \frac{1}{9}} = -\frac{1}{10}$$

Infinite Geometric Series

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	-	-	1-

Evaluate the following sums.

$$(1) \sum_{n=1}^{\infty} \left[\left(\frac{1}{2} \right)^n \sin \frac{n\pi}{3} \right]$$

$$= \frac{1}{2} \sin \frac{\pi}{3} + \frac{1}{2^2} \sin \frac{2\pi}{3} + \frac{1}{2^3} \sin \pi + \frac{1}{2^4} \sin \frac{4\pi}{3} + \dots$$

$$= \frac{\sqrt{3}}{2^2} + \frac{\sqrt{3}}{2^3} - \frac{\sqrt{3}}{2^5} - \frac{\sqrt{3}}{2^6} + \frac{\sqrt{3}}{2^8} + \frac{\sqrt{3}}{2^9} - \dots$$

$$= \frac{\sqrt{3}}{2^2} \left(1 - \frac{1}{2^3} + \frac{1}{2^6} - \dots \right) + \frac{\sqrt{3}}{2^3} \left(1 - \frac{1}{2^3} + \frac{1}{2^6} - \dots \right)$$

$$= \frac{\sqrt{3}}{4} \cdot \frac{1}{1 + \frac{1}{8}} + \frac{\sqrt{3}}{8} \cdot \frac{1}{1 + \frac{1}{8}}$$

$$= \frac{2\sqrt{3}}{9} + \frac{\sqrt{3}}{9}$$

$$= \frac{\sqrt{3}}{3}$$

N 115 b

$$(2) \sum_{n=1}^{\infty} \left[\left(\frac{1}{2} \right)^n \cos \frac{2n\pi}{3} \right]$$

$$= \frac{1}{2} \cos \frac{2\pi}{3} + \frac{1}{2^2} \cos \frac{4\pi}{3} + \frac{1}{2^3} \cos 2\pi + \frac{1}{2^4} \cos \frac{8\pi}{3} + \dots$$

$$= -\frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^3} - \frac{1}{2^5} - \frac{1}{2^6} + \frac{1}{2^6} - \dots$$

$$= -\frac{1}{2^2} \left(1 + \frac{1}{2^3} + \frac{1}{2^6} + \dots \right) - \frac{1}{2^3} \left(1 + \frac{1}{2^3} + \frac{1}{2^6} + \dots \right) + \frac{1}{2^3} \left(1 + \frac{1}{2^3} + \frac{1}{2^6} + \dots \right)$$

$$= -\frac{1}{4} \cdot \frac{1}{1 - \frac{1}{8}} - \frac{1}{8} \cdot \frac{1}{1 - \frac{1}{8}} + \frac{1}{8} \cdot \frac{1}{1 - \frac{1}{8}}$$

$$= -\frac{1}{4} \cdot \frac{8}{7}$$

$$= -\frac{2}{7}$$

Infinite Geometric Series

Time : to : Date Name

100%	90%	80%	70%	60%~
(mistakes) 0	-	-	1	2-

1. Given an infinite geometric series whose sum is 9 and whose second term is -4 , determine the first term and the common ratio of the series.

[Sol] Letting a be the first term, and letting r be the common ratio,

$$\begin{cases} \frac{a}{1-r} = 9 & \dots \textcircled{1} \\ ar = -4 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{2}$, $a = -\frac{4}{r}$. Substituting this into $\textcircled{1}$ and simplifying,

$$9r^2 - 9r - 4 = 0$$

$$(3r+1)(3r-4) = 0$$

$$\therefore r = -\frac{1}{3}, \frac{4}{3}$$

Since $|r| < 1$, $r = -\frac{1}{3}$. Substituting this value into $\textcircled{2}$, $a = 12$.

Therefore, the first term is 12, and the common ratio is $-\frac{1}{3}$.

2. Given an infinite geometric series whose sum is 3, and given another series each of whose terms is the square of the first series' terms, and whose sum is 6, determine the first term and the common ratio of the first series.

[Sol] Letting a be the first term, and letting r be the common ratio of the first series,

$$\begin{cases} \frac{a}{1-r} = 3 & \dots \textcircled{1} \\ \frac{a^2}{1-r^2} = 6 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$, $a = 3(1-r)$. Substituting this into $\textcircled{2}$ and simplifying,

$$5r^2 - 6r + 1 = 0$$

$$(5r-1)(r-1) = 0$$

$$\therefore r = \frac{1}{5}, 1$$

Since $|r| < 1$, $r = \frac{1}{5}$. Substituting this value into $\textcircled{1}$, $a = \frac{12}{5}$.

Therefore, the first term is $\frac{12}{5}$, and the common ratio is $\frac{1}{5}$.

3. In each of the following exercises, determine the range of values of x at which the given infinite geometric series converges, and then find its sum.

Ex. $1 + 2x + 4x^2 + 8x^3 + \dots$

[Sol] Since the common ratio $r = 2x$,

The series converges when $|2x| < 1$. $\therefore |x| < \frac{1}{2}$

Therefore, in this case: $S = \frac{1}{1-2x}$

(1) $x + x^2(x-2) + x^3(x-2)^2 + \dots$

[Sol] Since the common ratio $r = x(x-2)$,

The series converges when $|x(x-2)| < 1$.

$$\therefore x^2 - 2x - 1 < 0 \text{ or } x^2 - 2x + 1 > 0$$

$$\therefore 1 - \sqrt{2} < x < 1, \quad 1 < x < 1 + \sqrt{2}$$

$$\begin{aligned} \text{Therefore, in this case: } S &= \frac{x}{1-x(x-2)} \\ &= \frac{x}{1+2x-x^2} \end{aligned}$$

(2) $x + \frac{2x^2}{1+x} + \frac{4x^3}{(1+x)^2} + \dots$

[Sol] Since the common ratio $r = \frac{2x}{1+x}$,

The series converges when $\left| \frac{2x}{1+x} \right| < 1 \dots \textcircled{1}$

$$\text{When } -1 < \frac{2x}{1+x},$$

$$\frac{1+3x}{1+x} > 0$$

$$\therefore x < -1, x > -\frac{1}{3}$$

$$\text{When } \frac{2x}{1+x} < 1,$$

$$\frac{x-1}{1+x} < 0$$

$$\therefore -1 < x < 1$$

The solution to $\textcircled{1}$ is $-\frac{1}{3} < x < 1$.

$$\text{Therefore, in this case: } S = \frac{x}{1 - \frac{2x}{1+x}} = \frac{x(1+x)}{1-x}$$

Infinite Geometric Series

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	-	-	-

1. Determine the range of values of the real number x at which the given infinite geometric series converges, and then find the sum.

$$x + x(2 - x^2) + x(2 - x^2)^2 + \dots + x(2 - x^2)^n + \dots$$

[Sol]

- (a) When $x \neq 0$, ... ①

Since the common ratio is $2 - x^2$,

The series converges when $-1 < 2 - x^2 < 1$.

When $-1 < 2 - x^2$,

$$-\sqrt{3} < x < \sqrt{3} \quad \dots ②$$

When $2 - x^2 < 1$,

$$x < -1, x > 1 \quad \dots ③$$

From ①, ② and ③, $-\sqrt{3} < x < -1$, $1 < x < \sqrt{3}$

Therefore, in this case: $S = \frac{x}{1 - (2 - x^2)} = \frac{x}{x^2 - 1} \quad \dots ④$

- (b) When $x = 0$,

The series becomes $0 + 0 + 0$, which converges to sum $S = 0$.

This is the same result obtained by substituting $x = 0$ into ④ above.

Therefore, from (a) and (b),

The series converges when: $-\sqrt{3} < x < -1$, $x = 0$, $1 < x < \sqrt{3}$

The sum is: $S = \frac{x}{x^2 - 1}$

N 117 b

2. Determine the range of values of the real number x at which the given infinite geometric series converges, and then find the sum.

$$(1-x) + (1-x)x(3-4x) + (1-x)x^2(3-4x)^2 + \dots + (1-x)x^{n-1}(3-4x)^{n-1} + \dots$$

[Sol] The series converges when $-1 < x(3-4x) < 1$.

When $-1 < x(3-4x)$,

$$4x^2 - 3x - 1 < 0$$

$$(4x+1)(x-1) < 0$$

$$\therefore -\frac{1}{4} < x < 1$$

When $x(3-4x) < 1$,

$$4x^2 - 3x + 1 = \left(2x - \frac{3}{4}\right)^2 + \frac{7}{16} > 0,$$

This holds true for all real numbers x .

Additionally, when $x = 1$, the sum is $S = 0$ which indicates convergence.

Therefore, the series converges when $-\frac{1}{4} < x \leq 1$.

$$\begin{aligned} \text{The sum is: } S &= \frac{1-x}{1-x(3-4x)} \\ &= -\frac{x-1}{4x^2-3x+1} \end{aligned}$$

Infinite Geometric Series

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	-	-	1-

1. There is a certain infinite geometric series whose first term and common ratio are both real numbers and the sum of whose terms is 2. The sum of another series, each of whose terms is the square of the first series' terms, is 1. Obtain the common ratio of the first series.

[Sol] Letting a be the first term and letting r be the common ratio of the first series,

Since the sum is 2,

$$\frac{a}{1-r} = 2 \quad \dots \textcircled{1}$$

$$-1 < r < 1 \quad \dots \textcircled{2}$$

Letting a^2 be the first term and r^2 be the common ratio of the second series,

Since the sum is 1,

$$\frac{a^2}{1-r^2} = 1 \quad \dots \textcircled{3}$$

$$0 \leq r^2 < 1 \quad \dots \textcircled{4}$$

From $\textcircled{1}$, $a = 2(1-r)$ $\dots \textcircled{5}$

From $\textcircled{3}$, $a^2 = 1-r^2$ $\dots \textcircled{6}$

Substituting $\textcircled{5}$ into $\textcircled{6}$,

$$4(1-r)^2 = 1-r^2$$

$$\therefore (5r-3)(r-1) = 0$$

$$r = \frac{3}{5}, 1$$

From $\textcircled{2}$, $r = \frac{3}{5}$

Therefore, the common ratio of the first series is $\frac{3}{5}$.

N 118 b

2. There is a certain infinite geometric series whose first term and common ratio are both real numbers and the sum of whose terms is 2. The sum of another series, each of whose terms is the cube of the first series' terms, is 1. Obtain the common ratio of the first series.

[Sol] Letting a be the first term and letting r be the common ratio of the first series,

Since the sum is 2,

$$\frac{a}{1-r} = 2 \quad \dots \textcircled{1} \qquad -1 < r < 1 \quad \dots \textcircled{2}$$

Letting a^3 be the first term and r^3 be the common ratio of the second series,

Since the sum is 1,

$$\frac{a^3}{1-r^3} = 1 \quad \dots \textcircled{3} \qquad -1 < r^3 < 1 \quad \dots \textcircled{4}$$

$$\text{From } \textcircled{1}, a = 2(1-r) \quad \dots \textcircled{5}$$

$$\text{From } \textcircled{3}, a^3 = 1 - r^3 \quad \dots \textcircled{6}$$

Substituting $\textcircled{5}$ into $\textcircled{6}$,

$$8(1-r)^3 = 1 - r^3$$

$$\therefore (1-r)(7r^2 - 17r + 7) = 0$$

$$r = 1, \frac{17 \pm \sqrt{93}}{14}$$

$$\text{From } \textcircled{2}, r = \frac{17 - \sqrt{93}}{14}$$

Therefore, the common ratio of the first series is $\frac{17 - \sqrt{93}}{14}$.

Infinite Geometric Series

Time : to : Date Name

100%	90%	80%	70%	60%~
(mistakes) 0	-	1	2	3-

1. Convert
- $x = 0.\overline{36} = 0.363636 \dots$
- into a fraction.

[Method 1]

$$x = 0.36 + 0.0036 + 0.000036 + \dots$$

This is an infinite geometric series with first term $\boxed{0.36}$ and common ratio $\boxed{0.01}$.

$$\begin{aligned}\therefore x &= \frac{\boxed{0.36}}{1 - \boxed{0.01}} \\ &= \frac{\boxed{4}}{\boxed{11}}\end{aligned}$$

[Method 2]

$$x = 0.363636 \dots$$

Multiplying both sides by 100,

$$100x = 36.363636 \dots$$

$$100x - x = \boxed{36}$$

$$\therefore x = \frac{\boxed{4}}{\boxed{11}}$$

2. Convert
- $x = 0.\overline{12}$
- into a fraction by using the two methods shown above.

[Method 1]

$$x = 0.12 + 0.0012 + 0.000012 + \dots$$

This is an infinite geometric series with first term 0.12 and common ratio 0.01.

$$\therefore x = \frac{0.12}{1 - 0.01} = \frac{12}{99} = \frac{4}{33}$$

[Method 2]

$$x = 0.121212 \dots$$

Multiplying both sides by 100,

$$100x = 12.121212 \dots$$

$$100x - x = 12$$

$$\therefore x = \frac{12}{99} = \frac{4}{33}$$

3. Convert
- $x = 0.\overline{324} = 0.3242424 \dots$
- into a fraction by using the two methods shown above.

[Method 1]

$$x = 0.3 + 0.024 + 0.00024 + 0.0000024 + \dots$$

$$= 0.3 + \frac{0.024}{1 - 0.01}$$

$$= \frac{3}{10} + \frac{24}{990}$$

$$= \frac{3}{10} + \frac{4}{165} = \frac{107}{330}$$

[Method 2]

$$x = 0.3242424 \dots$$

Multiplying both sides by 100,

$$100x = 32.4242424 \dots$$

$$100x - x = 32.1$$

$$\therefore x = \frac{321}{990} = \frac{107}{330}$$

N 119 b

4. Convert the following repeating decimals into fractions.

$$(1) \quad 0.\overline{63} = \frac{0.63}{1 - 0.01} = \frac{63}{99} = \frac{7}{11}$$

$$(2) \quad 0.2\overline{18} = 0.2 + \frac{0.018}{1 - 0.01} = \frac{1}{5} + \frac{18}{990} = \frac{12}{55}$$

$$(3) \quad 5.\overline{477} = 5 + \frac{0.477}{1 - 0.001} = 5 + \frac{477}{999} = \frac{608}{111} \quad \left[= 5\frac{53}{111} \right]$$

$$(4) \quad 0.\overline{185} = \frac{0.185}{1 - 0.001} = \frac{185}{999} = \frac{5}{27}$$

$$\begin{aligned}(5) \quad 0.03\overline{18} &= \frac{3}{100} + \frac{18}{9900} \\ &= \frac{3}{100} + \frac{1}{550} \\ &= \frac{35}{1100} = \frac{7}{220}\end{aligned}$$

Infinite Geometric Series

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	-	1	2-

1. Determine the sum, S , of each of the following infinite geometric series.

(1) $16 + 8\sqrt{2} + 8 + \dots$

[Sol] The first term is 16. The common ratio is $\frac{\sqrt{2}}{2}$.

$$S = \frac{16}{1 - \frac{\sqrt{2}}{2}} = \frac{32(2 + \sqrt{2})}{(2 - \sqrt{2})(2 + \sqrt{2})} = 16(2 + \sqrt{2})$$

(2) $x^2 + \frac{x^2}{1 + 3x^4} + \frac{x^2}{(1 + 3x^4)^2} + \dots$

[Sol] (a) When $x \neq 0$, $\left| \frac{1}{1 + 3x^4} \right| < 1$

Therefore, the series converges, and the sum is:

$$S = \frac{x^2}{1 - \frac{1}{1 + 3x^4}} = \frac{1 + 3x^4}{3x^2}$$

(b) When $x = 0$, each term becomes 0.

Therefore, $S = 0$.

2. Determine the range of values of x at which the given infinite geometric series converges, and then find the sum.

$$2 + 2x(x - 3) + 2x^2(x - 3)^2 + \dots$$

[Sol] Since the common ratio $r = x(x - 3)$,

The series converges when $|x(x - 3)| < 1$.

$$\therefore x^2 - 3x - 1 < 0 \text{ or } x^2 - 3x + 1 > 0$$

$$\therefore \frac{3 - \sqrt{13}}{2} < x < \frac{3 - \sqrt{5}}{2}, \quad \frac{3 + \sqrt{5}}{2} < x < \frac{3 + \sqrt{13}}{2}$$

Therefore, in this case: $S = \frac{2}{1 - x(x - 3)}$

$$= \frac{2}{x^2 - 3x - 1}$$

N 120 b

3. Evaluate the following sum.

$$\begin{aligned}
 & \sum_{n=1}^{\infty} \left[\left(-\frac{2}{3} \right)^n \cos \frac{2n\pi}{3} \right] \\
 &= \left(-\frac{2}{3} \right) \cos \frac{2\pi}{3} + \left(-\frac{2}{3} \right)^2 \cos \frac{4\pi}{3} + \left(-\frac{2}{3} \right)^3 \cos 2\pi + \left(-\frac{2}{3} \right)^4 \cos \frac{8\pi}{3} + \left(-\frac{2}{3} \right)^5 \cos \frac{10\pi}{3} + \dots \\
 &= -\frac{1}{2} \left(-\frac{2}{3} \right) - \frac{1}{2} \left(-\frac{2}{3} \right)^2 + \left(-\frac{2}{3} \right)^3 - \frac{1}{2} \left(-\frac{2}{3} \right)^4 - \frac{1}{2} \left(-\frac{2}{3} \right)^5 + \dots \\
 &= -\frac{1}{2} \cdot \left(-\frac{2}{3} \right) \left[1 + \left(-\frac{2}{3} \right)^3 + \dots \right] - \frac{1}{2} \left(-\frac{2}{3} \right)^2 \left[1 + \left(-\frac{2}{3} \right)^3 + \dots \right] + \left(-\frac{2}{3} \right)^3 \left[1 + \left(-\frac{2}{3} \right)^3 + \dots \right] \\
 &= \frac{1}{3} \cdot \frac{1}{1 - \left(-\frac{2}{3} \right)^3} - \frac{2}{9} \cdot \frac{1}{1 - \left(-\frac{2}{3} \right)^3} - \frac{8}{27} \cdot \frac{1}{1 - \left(-\frac{2}{3} \right)^3} \\
 &= \frac{1}{3} \cdot \frac{27}{35} - \frac{2}{9} \cdot \frac{27}{35} - \frac{8}{27} \cdot \frac{27}{35} \\
 &= -\frac{1}{7}
 \end{aligned}$$

Infinite Series

Time : to : Date Name

100%	90%	80%	70%	60%~
(mistakes) 0	-	-	-	1-

A general infinite series, $\{a_n\}$ of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots \quad \text{--- ①}$$

is called an *Infinite Series* and is expressed as $\sum_{n=1}^{\infty} a_n$.

The partial sum of the first n terms is expressed as $S_n = \sum_{k=1}^n a_k$.

When the sequence $S_1, S_2, S_3, \dots, S_n, \dots$ converges to a value S , then the infinite series ① is said to **converge**, and the sum is S :

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = S$$

When $\{S_n\}$ diverges, the infinite series ① is said to **diverge**.

Determine whether each of the following infinite series converges or diverges. If it converges, determine the sum.

$$(1) \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} + \cdots$$

[Sol] The n^{th} term is: $a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \boxed{\frac{1}{n+1}}$

Expressing the partial sum of the first n terms,

$$\begin{aligned} S_n &= \sum_{k=1}^n \frac{1}{k(k+1)} \\ &= \sum_{k=1}^n \left(\frac{1}{k} - \boxed{\frac{1}{k+1}} \right) \\ &= \left(\frac{1}{1} - \boxed{\frac{1}{2}} \right) + \left(\frac{1}{2} - \boxed{\frac{1}{3}} \right) + \left(\frac{1}{3} - \boxed{\frac{1}{4}} \right) + \cdots + \left(\frac{1}{n} - \boxed{\frac{1}{n+1}} \right) \\ &= 1 - \boxed{\frac{1}{n+1}} \end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(1 - \boxed{\frac{1}{n+1}} \right) = \boxed{1}$$

Therefore, the series converges, and the sum is $\boxed{1}$.

N 121 b

$$(2) \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots + \frac{1}{(2n-1)(2n+1)} + \cdots$$

[Sol] The n^{th} term is: $a_n = \frac{1}{(2n-1)(2n+1)}$

Breaking this up into separate fractions,

$$\frac{1}{(2n-1)(2n+1)} = \frac{A}{2n-1} - \frac{B}{2n+1}$$

Solving for A and B ,

$$A = B = \frac{1}{2}$$

$$\therefore a_n = \frac{1}{2} \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right)$$

Expressing the partial sum of the first n terms,

$$\begin{aligned} S_n &= \sum_{k=1}^n \left[\frac{1}{2} \left(\frac{1}{2k-1} - \frac{1}{2k+1} \right) \right] \\ &= \frac{1}{2} \left[\left(\frac{1}{1} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \left(\frac{1}{5} - \frac{1}{7} \right) + \cdots \right. \\ &\quad \left. \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right) \right] \\ &= \frac{1}{2} \left(1 - \frac{1}{2n+1} \right) \end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left[\frac{1}{2} \left(1 - \frac{1}{2n+1} \right) \right] = \frac{1}{2}$$

Therefore, the series converges, and the sum is $\frac{1}{2}$.

$$(3) 1 + 2 + 3 + \cdots + n + \cdots$$

[Sol] Expressing the partial sum of the first n terms,

$$S_n = \frac{1}{2}n(n+1)$$

$$\lim_{n \rightarrow \infty} S_n = \infty$$

Therefore, the series diverges.

Infinite Series

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	-	1	2

1. Determine the sum of each of the following infinite series.

$$(1) \frac{1}{1^2+1} + \frac{1}{2^2+2} + \frac{1}{3^2+3} + \cdots + \frac{1}{n^2+n} + \cdots$$

[Sol] The n^{th} term is:

$$a_n = \frac{1}{n^2+n} = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$S_n = \sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{k+1} \right)$$

$$= \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

$$= 1 - \frac{1}{n+1}$$

$$\therefore S = \lim_{n \rightarrow \infty} S_n = 1$$

$$(2) \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \cdots + \frac{1}{n(n+1)(n+2)} + \cdots$$

(Hint: Using the equality $\frac{1}{n(n+1)(n+2)} = \frac{A}{n(n+1)} - \frac{B}{(n+1)(n+2)}$,
begin by solving for A and B .)

$$\begin{aligned} \text{[Sol]} \text{ The } n^{\text{th}} \text{ term is: } a_n &= \frac{1}{n(n+1)(n+2)} = \frac{A}{n(n+1)} - \frac{B}{(n+1)(n+2)} \\ &= \frac{n(A-B) + 2A}{n(n+1)(n+2)} \end{aligned}$$

From the above,

$$A = B \text{ and } 2A = 1.$$

$$\text{Therefore, } A = B = \frac{1}{2}.$$

$$S_n = \sum_{k=1}^n \frac{1}{2} \left[\frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)} \right]$$

$$= \frac{1}{2} \left[\frac{1}{1 \cdot 2} - \frac{1}{(n+1)(n+2)} \right]$$

$$\therefore S = \lim_{n \rightarrow \infty} S_n = \frac{1}{4}$$

N 122 b

2. In each exercise, state whether the given series converges or diverges.
If it converges, determine the sum.

$$(1) \frac{1}{\sqrt{2}+1} + \frac{1}{\sqrt{3}+\sqrt{2}} + \frac{1}{\sqrt{4}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{n+1}+\sqrt{n}} + \cdots$$

[Sol] The n^{th} term is:

$$a_n = \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

Rationalizing the RHS,

$$a_n = \sqrt{n+1} - \sqrt{n}$$

$$\begin{aligned} S_n &= \sum_{k=1}^n (\sqrt{k+1} - \sqrt{k}) \\ &= (\sqrt{2} - 1) + (\sqrt{3} - \sqrt{2}) + (\sqrt{4} - \sqrt{3}) + \cdots + (\sqrt{n+1} - \sqrt{n}) \\ &= \sqrt{n+1} - 1 \end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \infty$$

Therefore, the series diverges.

$$(2) \frac{1}{\sqrt{2}+2} + \frac{1}{2\sqrt{3}+3\sqrt{2}} + \cdots + \frac{1}{n\sqrt{n+1} + (n+1)\sqrt{n}} + \cdots$$

[Sol] The n^{th} term is:

$$a_n = \frac{1}{n\sqrt{n+1} + (n+1)\sqrt{n}}$$

Rationalizing the RHS,

$$a_n = \frac{(n+1)\sqrt{n} - n\sqrt{n+1}}{n(n+1)} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}$$

$$S_n = \sum_{k=1}^n \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}} \right) = 1 - \frac{1}{\sqrt{n+1}}$$

$$\therefore \lim_{n \rightarrow \infty} S_n = 1$$

Therefore, the series converges, and the sum is 1.

Infinite Series

Time : to : Date Name

100%	90%	80%	70%	60%
(minutes) 0	-	-	-	-

In each exercise, state whether the given series converges or diverges.
If it converges, determine the sum.

$$(1) \frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \cdots + \frac{1}{1+2+\cdots+n} + \cdots$$

[Sol] The n^{th} term is:

$$a_n = \frac{1}{1+2+\cdots+n} = \frac{1}{\frac{1}{2}n(n+1)} = 2\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$\begin{aligned} S_n &= \sum_{k=1}^n \left[2\left(\frac{1}{k} - \frac{1}{k+1}\right) \right] = 2\left[\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)\right] \\ &= 2\left(1 - \frac{1}{n+1}\right) \end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left[2\left(1 - \frac{1}{n+1}\right) \right] = 2$$

Therefore, the series converges, and the sum is 2.

$$(2) \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 + 2 \cdot 3} + \cdots + \frac{1}{1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1)} + \cdots$$

[Sol] $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1)$

$$\begin{aligned} &= \sum_{k=1}^n [k(k+1)] = \sum_{k=1}^n k^2 + \sum_{k=1}^n k = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \\ &= \frac{n(n+1)(n+2)}{3} \end{aligned}$$

Therefore, the n^{th} term is:

$$a_n = \frac{3}{n(n+1)(n+2)} = \frac{3}{2} \left[\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)} \right]$$

$$S_n = \sum_{k=1}^n \left\{ \frac{3}{2} \left[\frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)} \right] \right\} = \frac{3}{2} \left[\frac{1}{2} - \frac{1}{(n+1)(n+2)} \right]$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \frac{3}{4}$$

Thus, the series converges, and the sum is $\frac{3}{4}$.

N 123 b

$$(3) \frac{4}{1 \cdot 2 \cdot 3} + \frac{5}{2 \cdot 3 \cdot 4} + \frac{6}{3 \cdot 4 \cdot 5} + \dots$$

$$\begin{aligned} \text{[Sol]} \text{ The } n^{\text{th}} \text{ term is: } a_n &= \frac{n+3}{n(n+1)(n+2)} = \frac{A}{n(n+1)} + \frac{B}{(n+1)(n+2)} \\ &= \frac{(A+B)n+2A}{n(n+1)(n+2)} \end{aligned}$$

Comparing numerators,

$$n+3 = (A+B)n+2A$$

$$\text{Therefore, } A = \frac{3}{2}, \quad B = -\frac{1}{2}$$

$$\begin{aligned} S_n &= \sum_{k=1}^n \left[\frac{3}{2} \cdot \frac{1}{k(k+1)} - \frac{1}{2} \cdot \frac{1}{(k+1)(k+2)} \right] \\ &= \sum_{k=1}^n \left[\frac{3}{2} \left(\frac{1}{k} - \frac{1}{k+1} \right) \right] - \sum_{k=1}^n \left[\frac{1}{2} \left(\frac{1}{k+1} - \frac{1}{k+2} \right) \right] \\ &= \frac{3}{2} \left(1 - \frac{1}{n+1} \right) - \frac{1}{2} \left(\frac{1}{2} - \frac{1}{n+2} \right) \end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \frac{3}{2} - \frac{1}{4} = \frac{5}{4}$$

Thus, the series converges, and the sum is $\frac{5}{4}$.

Infinite Series

Time : to : Date Name

100% (mistakes) 0	90%	80%	70%	60%

Given that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge, when $\sum_{n=1}^{\infty} a_n = \alpha$ and $\sum_{n=1}^{\infty} b_n = \beta$, then,

- $\sum_{n=1}^{\infty} (ka_n) = k\alpha$ (where k is a constant)
- $\sum_{n=1}^{\infty} (a_n + b_n) = \alpha + \beta$
- $\sum_{n=1}^{\infty} (ha_n + kb_n) = h\alpha + k\beta$ (where h and k are constants)

Determine the sum of each given series.

$$\begin{aligned}(1) \quad \sum_{n=1}^{\infty} \frac{3}{4^{n-2}} &= \sum_{n=1}^{\infty} \frac{12}{4^{n-1}} \\ &= \frac{12}{1 - \frac{1}{4}} \\ &= 16\end{aligned}$$

$$\begin{aligned}(2) \quad \sum_{n=1}^{\infty} \frac{1}{2 \cdot 5^n} &= \frac{\frac{1}{10}}{1 - \frac{1}{5}} \\ &= \frac{\frac{1}{10}}{\frac{4}{5}} = \frac{1}{8}\end{aligned}$$

$$\begin{aligned}(3) \quad \sum_{n=1}^{\infty} \left(\frac{1}{3^{n-1}} + \frac{2^n}{3^n} \right) &= \sum_{n=1}^{\infty} \frac{1}{3^{n-1}} + \sum_{n=1}^{\infty} \left(\frac{2}{3} \right)^n \\ &= \frac{1}{1 - \frac{1}{3}} + \frac{\frac{2}{3}}{1 - \frac{2}{3}} = \frac{3}{2} + 2 = \frac{7}{2}\end{aligned}$$

N 124 b

$$\begin{aligned}
 (4) \quad \sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} - \frac{1}{3^n} \right) &= \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{3^n} \\
 &= \frac{1}{1 - \frac{1}{2}} - \frac{\frac{1}{3}}{1 - \frac{1}{3}} = 2 - \frac{1}{2} = \frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 (5) \quad \sum_{n=1}^{\infty} \frac{2^n - 3^n}{4^n} &= \sum_{n=1}^{\infty} \left(\frac{1}{2} \right)^n - \sum_{n=1}^{\infty} \left(\frac{3}{4} \right)^n \\
 &= \frac{\frac{1}{2}}{1 - \frac{1}{2}} - \frac{\frac{3}{4}}{1 - \frac{3}{4}} = 1 - 3 = -2
 \end{aligned}$$

$$\begin{aligned}
 (6) \quad \sum_{n=1}^{\infty} \frac{3^n - 4^n}{5^{n-1}} &= 5 \cdot \sum_{n=1}^{\infty} \left(\frac{3}{5} \right)^n - 5 \cdot \sum_{n=1}^{\infty} \left(\frac{4}{5} \right)^n \\
 &= 5 \cdot \frac{\frac{3}{5}}{1 - \frac{3}{5}} - 5 \cdot \frac{\frac{4}{5}}{1 - \frac{4}{5}} = 5 \left(\frac{3}{2} - 4 \right) = -\frac{25}{2}
 \end{aligned}$$

Infinite Series

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	-	-	1-

Determine the sum of each of the following infinite series.

$$(1) \sum_{k=1}^{\infty} \left(\cos \frac{\pi}{k} - \cos \frac{\pi}{k+1} \right)$$

$$[\text{Sol}] \text{ Letting } \sum_{k=1}^{\infty} \left(\cos \frac{\pi}{k} - \cos \frac{\pi}{k+1} \right) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\cos \frac{\pi}{k} - \cos \frac{\pi}{k+1} \right).$$

$$\begin{aligned} \text{Letting } S_n &= \sum_{k=1}^n \left(\cos \frac{\pi}{k} - \cos \frac{\pi}{k+1} \right), \\ &= \left(\cos \pi - \cos \frac{\pi}{2} \right) + \left(\cos \frac{\pi}{2} - \cos \frac{\pi}{3} \right) + \cdots + \left(\cos \frac{\pi}{n} - \cos \frac{\pi}{n+1} \right) \\ &= \cos \pi - \cos \frac{\pi}{n+1} \\ &= -1 - \cos \frac{\pi}{n+1} \end{aligned}$$

$$\therefore \sum_{k=1}^{\infty} \left(\cos \frac{\pi}{k} - \cos \frac{\pi}{k+1} \right) = \lim_{n \rightarrow \infty} S_n = -2$$

N 125 b

$$(2) \sum_{k=1}^{\infty} \left(2 \cos \frac{3\pi}{2^{k+1}} \sin \frac{\pi}{2^{k+1}} \right)$$

$$[\text{Sol}] \text{ Since } \sum_{k=1}^{\infty} \left(2 \cos \frac{3\pi}{2^{k+1}} \sin \frac{\pi}{2^{k+1}} \right) = \sum_{k=1}^{\infty} \left(\sin \frac{4\pi}{2^{k+1}} - \sin \frac{2\pi}{2^{k+1}} \right),$$

$$\text{Letting } \sum_{k=1}^{\infty} \left(\sin \frac{4\pi}{2^{k+1}} - \sin \frac{2\pi}{2^{k+1}} \right) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\sin \frac{4\pi}{2^{k+1}} - \sin \frac{2\pi}{2^{k+1}} \right),$$

$$\text{Letting } S_n = \sum_{k=1}^n \left(\sin \frac{4\pi}{2^{k+1}} - \sin \frac{2\pi}{2^{k+1}} \right),$$

$$= \sum_{k=1}^n \left(\sin \frac{\pi}{2^{k-1}} - \sin \frac{\pi}{2^k} \right)$$

$$= \left(\sin \pi - \sin \frac{\pi}{2} \right) + \left(\sin \frac{\pi}{2} - \sin \frac{\pi}{2^2} \right) + \cdots + \left(\sin \frac{\pi}{2^{n-1}} - \sin \frac{\pi}{2^n} \right)$$

$$= \sin \pi - \sin \frac{\pi}{2^n}$$

$$= 0 - \sin \frac{\pi}{2^n}$$

$$\therefore \sum_{k=1}^{\infty} \left(2 \cos \frac{3\pi}{2^{k+1}} \sin \frac{\pi}{2^{k+1}} \right) = \lim_{n \rightarrow \infty} S_n = 0$$

Infinite Series

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	-	-	1-

1. Given that $a_n = \frac{1}{n+1} - \frac{2}{n+2} + \frac{1}{n+3}$, determine the sum of the series $\sum_{n=1}^{\infty} a_n$.

$$\begin{aligned} \text{[Sol]} \quad a_n &= \frac{1}{n+1} - \frac{2}{n+2} + \frac{1}{n+3} \\ &= \left(\frac{1}{n+1} - \frac{1}{n+2} \right) - \left(\frac{1}{n+2} - \frac{1}{n+3} \right) \end{aligned}$$

$$\begin{aligned} \text{Letting } S_n &= \sum_{k=1}^n \left[\left(\frac{1}{k+1} - \frac{1}{k+2} \right) - \left(\frac{1}{k+2} - \frac{1}{k+3} \right) \right] \\ &= \sum_{k=1}^n \left(\frac{1}{k+1} - \frac{1}{k+2} \right) - \sum_{k=1}^n \left(\frac{1}{k+2} - \frac{1}{k+3} \right) \\ &= \frac{1}{2} - \frac{1}{n+2} - \left(\frac{1}{3} - \frac{1}{n+3} \right) \\ &= \frac{1}{6} - \frac{1}{n+2} + \frac{1}{n+3} \end{aligned}$$

$$\therefore \sum_{n=1}^{\infty} a_n = \lim_{n \rightarrow \infty} S_n = \frac{1}{6}$$

N 126 b

2. Given the series $\sum_{k=1}^n \frac{k+1}{k^2(k+2)^2}$ (where $n \geq 2$), complete the following exercises.

(1) Determine the sum, S_n , of the series.

[Sol] Expressing $\frac{k+1}{k^2(k+2)^2}$ as $\frac{A}{k^2} - \frac{B}{(k+2)^2}$,

$$A(k+2)^2 - Bk^2 = k+1$$

$$(A-B)k^2 + (4A-1)(k+1) = 0$$

$$\therefore A = B = \frac{1}{4}$$

$$\begin{aligned} S_n &= \sum_{k=1}^n \left\{ \frac{1}{4} \left[\frac{1}{k^2} - \frac{1}{(k+2)^2} \right] \right\} \\ &= \frac{1}{4} \left\{ \left(\frac{1}{1^2} - \frac{1}{3^2} \right) + \left(\frac{1}{2^2} - \frac{1}{4^2} \right) + \left(\frac{1}{3^2} - \frac{1}{5^2} \right) + \dots \dots \dots \right. \\ &\quad \left. \dots \dots \dots + \left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} \right] + \left[\frac{1}{n^2} - \frac{1}{(n+2)^2} \right] \right\} \\ &= \frac{1}{4} \left[1 + \frac{1}{2^2} - \frac{1}{(n+1)^2} - \frac{1}{(n+2)^2} \right] \\ &= \frac{1}{4} \left[\frac{5}{4} - \frac{1}{(n+1)^2} - \frac{1}{(n+2)^2} \right] \end{aligned}$$

(2) Evaluate the limit, $\lim_{n \rightarrow \infty} S_n$.

$$\begin{aligned} \text{[Sol]} \quad \lim_{n \rightarrow \infty} S_n &= \lim_{n \rightarrow \infty} \left\{ \frac{1}{4} \left[\frac{5}{4} - \frac{1}{(n+1)^2} - \frac{1}{(n+2)^2} \right] \right\} \\ &= \frac{5}{16} \end{aligned}$$

Infinite Series

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100%	90%	80%	70%	60%
(mistakes) 0	-	-	1	2

Given that the infinite series $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ converges, and the sum is S , the partial sum S_n can be expressed as:

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$

The sequence $\{S_n\}$, made up of the partial sums, converges to S .

Therefore,

$$\lim_{n \rightarrow \infty} S_n = \boxed{S}$$

Since the n^{th} term is: $a_n = S_n - S_{n-1}$, $\lim_{n \rightarrow \infty} S_{n-1} = \boxed{S}$

$$\begin{aligned}\text{Thus, } \lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} (S_n - S_{n-1}) = \lim_{n \rightarrow \infty} S_n - \lim_{n \rightarrow \infty} S_{n-1} \\ &= S - \boxed{S} \\ &= \boxed{0}\end{aligned}$$

Answers: S, S, S, 0

When an infinite series **converges**, $\longrightarrow \lim_{n \rightarrow \infty} a_n = 0$

When an infinite series **diverges**, $\begin{cases} \lim_{n \rightarrow \infty} a_n = 0 \\ \lim_{n \rightarrow \infty} a_n \neq 0 \end{cases}$

(Note: When $\lim_{n \rightarrow \infty} a_n = 0$, the series may either converge or diverge.)

N 127 b

In each exercise, state whether the given series converges or diverges.

If it converges, determine the sum.

(1) $1 - 1 + 1 - 1 + 1 - 1 + \dots$

[Sol] $S_{2n} = 0$

$$\therefore \lim_{n \rightarrow \infty} S_{2n} = 0$$

$$S_{2n-1} = 1$$

$$\therefore \lim_{n \rightarrow \infty} S_{2n-1} = 1$$

$$\lim_{n \rightarrow \infty} S_{2n} \neq \lim_{n \rightarrow \infty} S_{2n-1}$$

Therefore, the series **diverges**.

(2) $(1 - 1) + (1 - 1) + (1 - 1) + \dots$

[Sol] $S_n = 0 + 0 + 0 + \dots$
 $= 0$

$$\therefore \lim_{n \rightarrow \infty} S_n = 0$$

Therefore, the series converges, and the sum is 0.

(3) $1 + (-1 + 1) + (-1 + 1) + \dots$

[Sol] $S_n = 1 + 0 + 0 + \dots$
 $= 1$

$$\therefore \lim_{n \rightarrow \infty} S_n = 1$$

Therefore, the series converges, and the sum is 1.

(4) $1 + (-1 - 1) + (1 + 1 + 1) + (-1 - 1 - 1 - 1) + \dots$

[Sol] $S_n = 1 - 2 + 3 - 4 + \dots + (-1)^{n-1}n$

When n is even,

$$S_n = (1 - 2) + (3 - 4) + \dots + (n - 1 - n)$$

$$= -\frac{n}{2}$$

$$\therefore \lim_{n \rightarrow \infty} S_n = -\infty$$

When n is odd,

$$S_n = 1 + (-2 + 3) + (-4 + 5) + \dots + (-n + 1 + n)$$

$$= \frac{n+1}{2}$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \infty$$

Therefore, the series diverges.

Infinite Series

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	1	-	2

In each exercise, state whether the given series converges or diverges.

If it converges, determine the sum.

$$(1) \frac{1}{1} + \frac{2}{3} + \frac{3}{5} + \frac{4}{7} + \dots$$

$$[\text{Sol}] a_n = \frac{n}{2n-1} \quad \therefore \lim_{n \rightarrow \infty} a_n = \frac{1}{2} \neq 0$$

Therefore, the series diverges.

$$(2) 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \dots + \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+1} - \dots$$

[Sol]

$$S_{2m-1} = 1 + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \dots + \left(-\frac{1}{m} + \frac{1}{m}\right) = 1$$

$$S_{2m} = 1 + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \dots + \left(-\frac{1}{m} + \frac{1}{m}\right) - \frac{1}{m+1}$$

$$= 1 - \frac{1}{m+1}$$

$$\lim_{m \rightarrow \infty} S_{2m-1} = \lim_{m \rightarrow \infty} S_{2m} = 1 \quad \therefore S = 1$$

Therefore, the series converges, and the sum is 1.

$$(3) \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right) + \dots$$

[Sol] Expressing the partial sum,

$$S_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

$$S = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1}\right) = 1$$

Therefore, the series converges, and the sum is 1.

N 128 b

$$(4) \left(\frac{1}{2} - \frac{2}{3}\right) + \left(\frac{2}{3} - \frac{3}{4}\right) + \left(\frac{3}{4} - \frac{4}{5}\right) + \dots$$

$$\begin{aligned} [\text{Sol}] S_n &= \left(\frac{1}{2} - \frac{2}{3}\right) + \left(\frac{2}{3} - \frac{3}{4}\right) + \dots + \left(\frac{n}{n+1} - \frac{n+1}{n+2}\right) \\ &= \frac{1}{2} + \left(-\frac{2}{3} + \frac{2}{3}\right) + \left(-\frac{3}{4} + \frac{3}{4}\right) + \dots + \left(-\frac{n}{n+1} + \frac{n}{n+1}\right) - \frac{n+1}{n+2} \\ &= \frac{1}{2} - \frac{n+1}{n+2} \end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(\frac{1}{2} - \frac{n+1}{n+2}\right) = \lim_{n \rightarrow \infty} \left(\frac{1}{2} - \frac{1 + \frac{1}{n}}{1 + \frac{2}{n}}\right) = -\frac{1}{2}$$

Therefore, the series converges, and the sum is $-\frac{1}{2}$.

$$(5) 2 - \frac{3}{2} + \frac{3}{2} - \frac{4}{3} + \dots + \frac{n+1}{n} - \frac{n+2}{n+1} + \frac{n+2}{n+1} - \dots$$

$$\begin{aligned} [\text{Sol}] S_{2m} &= 2 + \left(-\frac{3}{2} + \frac{3}{2}\right) + \left(-\frac{4}{3} + \frac{4}{3}\right) + \dots + \left(-\frac{m+1}{m} + \frac{m+1}{m}\right) - \frac{m+2}{m+1} \\ &= 2 - \frac{m+2}{m+1} \end{aligned}$$

$$S_{2m-1} = 2 + \left(-\frac{3}{2} + \frac{3}{2}\right) + \left(-\frac{4}{3} + \frac{4}{3}\right) + \dots + \left(-\frac{m+1}{m} + \frac{m+1}{m}\right) = 2$$

$$\lim_{m \rightarrow \infty} S_{2m-1} = 2, \quad \lim_{m \rightarrow \infty} S_{2m} = \lim_{m \rightarrow \infty} \left(2 - \frac{1 + \frac{2}{m}}{1 + \frac{1}{m}}\right) = 1$$

Therefore, the series diverges.

Infinite Series

Time : to : Date Name

100%	90%	80%	70%	60%~
(mistakes) 0	—	—	—	—

In each exercise, state whether the given series converges or diverges.
If it converges, determine the sum.

$$(1) \frac{2}{3} - \frac{4}{5} + \frac{4}{5} - \frac{6}{7} + \frac{6}{7} - \frac{8}{9} + \dots$$

$$[\text{Sol}] S_{2m} = \sum_{k=1}^m \left(\frac{2k}{2k+1} - \frac{2k+2}{2k+3} \right) = \frac{2}{3} - \frac{2m+2}{2m+3}$$

$$S_{2m-1} = S_{2m} + \frac{2m+2}{2m+3} = \frac{2}{3}$$

$$\therefore \lim_{m \rightarrow \infty} S_{2m} = \lim_{m \rightarrow \infty} \left(\frac{2}{3} - \frac{2 + \frac{2}{m}}{2 + \frac{3}{m}} \right) = -\frac{1}{3}$$

$$\lim_{m \rightarrow \infty} S_{2m-1} = \frac{2}{3}$$

Therefore, the series diverges.

N 129 b

$$(2) 1 - \frac{2}{3} + 3 + \frac{1}{2} - \frac{4}{9} - \frac{3}{2} + \frac{1}{4} - \frac{8}{27} + \frac{3}{4} + \frac{1}{8} - \frac{16}{81} - \frac{3}{8} + \dots$$

$$[\text{Sol}] S_{3m} = \sum_{k=1}^m \left(\frac{1}{2}\right)^{k-1} - \left[\frac{2}{3}\right] \sum_{k=1}^m \left(\frac{2}{3}\right)^{k-1} + [3] \sum_{k=1}^m \left(-\frac{1}{2}\right)^{k-1}$$

$$= \frac{1 - \left(\frac{1}{2}\right)^m}{1 - \frac{1}{2}} - \frac{2}{3} \cdot \frac{1 - \left(\frac{2}{3}\right)^m}{1 - \frac{2}{3}} + 3 \cdot \frac{1 - \left(-\frac{1}{2}\right)^m}{1 + \frac{1}{2}}$$

$$= 2 \left[1 - \left(\frac{1}{2}\right)^m \right] - 2 \left[1 - \left(\frac{2}{3}\right)^m \right] + 2 \left[1 - \left(-\frac{1}{2}\right)^m \right]$$

$$\therefore \lim_{m \rightarrow \infty} S_{3m} = 2 - 2 + 2 = 2$$

$$S_{3m-1} = S_{3m} - 3 \left(-\frac{1}{2}\right)^{m-1}$$

$$S_{3m-2} = S_{3m-1} + \frac{2}{3} \left(\frac{2}{3}\right)^{m-1}$$

$$\therefore \lim_{m \rightarrow \infty} S_{3m} = \lim_{m \rightarrow \infty} S_{3m-1} = \lim_{m \rightarrow \infty} S_{3m-2} = 2$$

Therefore, the series converges, and the sum is 2.

Time : to : Date Name

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(mistakes) 0	—	—	1	2

In each exercise, state whether the given series converges or diverges.
If it converges, determine the sum.

$$(1) \frac{1}{2} + \frac{2}{3} + \cdots + \frac{n}{n+1} + \cdots$$

[Sol] Letting a_n be the n^{th} term,

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n}{n+1} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n}} = 1 \neq 0$$

Therefore, the series diverges.

$$(2) \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)(1+2x^2)} + \frac{x^2}{(1+2x^2)(1+3x^2)} + \cdots$$

[Sol]

(a) When $x \neq 0$,

$$\begin{aligned} \text{The } n^{\text{th}} \text{ term is: } a_n &= \frac{x^2}{[1 + (n-1)x^2](1 + nx^2)} \\ &= \frac{1}{1 + (n-1)x^2} - \frac{1}{1 + nx^2} \end{aligned}$$

$$\begin{aligned} S_n &= \sum_{k=1}^n \left[\frac{1}{1 + (k-1)x^2} - \frac{1}{1 + kx^2} \right] \\ &= \left(\frac{1}{1} - \frac{1}{1+x^2} \right) + \left(\frac{1}{1+x^2} - \frac{1}{1+2x^2} \right) + \cdots + \left[\frac{1}{1 + (n-1)x^2} - \frac{1}{1 + nx^2} \right] \\ &= 1 - \frac{1}{1 + nx^2} \quad \therefore S = \lim_{n \rightarrow \infty} S_n = 1 \end{aligned}$$

(b) When $x = 0$,

$$\begin{aligned} S &= 0 + 0 + 0 + \cdots \\ &= 0 \end{aligned}$$

From (a) and (b), the series converges and:

$$\begin{cases} \text{When } x \neq 0, & S = 1 \\ \text{When } x = 0, & S = 0 \end{cases}$$

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$$(3) \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \frac{1}{5} - \dots$$

$$[\text{Sol}] S_{2m} = \sum_{k=2}^{m+1} \left(\frac{1}{k} - \frac{1}{k+1} \right) = \frac{1}{2} - \frac{1}{m+2}$$

$$\therefore \lim_{m \rightarrow \infty} S_{2m} = \frac{1}{2}$$

$$S_{2m+1} = S_{2m} + \frac{1}{m+2}$$

$$= \left(\frac{1}{2} - \frac{1}{m+2} \right) + \frac{1}{m+2} = \frac{1}{2}$$

$$\therefore \lim_{m \rightarrow \infty} S_{2m+1} = \frac{1}{2}$$

$$\therefore S = \lim_{m \rightarrow \infty} S_{2m} = \lim_{m \rightarrow \infty} S_{2m+1} = \frac{1}{2}$$

Therefore, the series converges, and the sum is $\frac{1}{2}$.

$$(4) \left(2 - \frac{3}{2} \right) + \left(\frac{3}{2} - \frac{4}{3} \right) + \dots + \left(\frac{n+1}{n} - \frac{n+2}{n+1} \right) + \dots$$

$$[\text{Sol}] S_n = \left(2 - \frac{3}{2} \right) + \left(\frac{3}{2} - \frac{4}{3} \right) + \dots + \left(\frac{n+1}{n} - \frac{n+2}{n+1} \right)$$

$$= 2 + \left(-\frac{3}{2} + \frac{3}{2} \right) + \left(-\frac{4}{3} + \frac{4}{3} \right) + \dots + \left(-\frac{n+1}{n} + \frac{n+1}{n} \right) - \frac{n+2}{n+1}$$

$$= 2 - \frac{n+2}{n+1}$$

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(2 - \frac{1 + \frac{2}{n}}{1 + \frac{1}{n}} \right) = 1$$

Therefore, the series converges, and the sum is 1.

Limits of Functions 1

Time : to : Date Name

100%	90%	80%	70%	60%
mistakes: 0	1	2	3	4

Assuming that $\lim_{x \rightarrow a} f(x) = \alpha$ and $\lim_{x \rightarrow a} g(x) = \beta$, where α and β have finite values,

■ $\lim_{x \rightarrow a} [kf(x)] = k\alpha$ (where k is a constant)

■ $\lim_{x \rightarrow a} [f(x) + g(x)] = \alpha + \beta$

■ $\lim_{x \rightarrow a} [f(x)g(x)] = \alpha\beta$

■ $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\alpha}{\beta}$ ($\beta \neq 0$)

1. Find the limit value of each of the following functions.

Ex. $\lim_{x \rightarrow 1} (x^2 + 2x + 3) = (1)^2 + 2(1) + 3$
 $= 6$

(1) $\lim_{x \rightarrow 1} [(x+1)(x-1)(x+3)] = 0$

(2) $\lim_{x \rightarrow 1} (x^4 - 3x^3 + 2x^2 + 5x - 4) = 1$

(3) $\lim_{x \rightarrow -2} \frac{x-1}{(x-2)(x-3)} = -\frac{3}{20}$

(4) $\lim_{x \rightarrow -1} \frac{2x^2 + 3x - 2}{2x^2 + 1} = \lim_{x \rightarrow -1} \frac{(2x-1)(x+2)}{2x^2 + 1} = 0$

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2. Find the limit value of each of the following functions.

$$(1) \lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} = \lim_{x \rightarrow 3} \frac{(x - 3)(x + 3)}{x - 3} = \lim_{x \rightarrow 3} (x + 3) = \boxed{6}$$

$$(2) \lim_{x \rightarrow -5} \frac{x^2 + 2x - 15}{x + 5} = \lim_{x \rightarrow -5} \frac{(x + 5)(x - 3)}{x + 5} = \lim_{x \rightarrow -5} (x - 3) = -8$$

$$(3) \lim_{x \rightarrow 0} \frac{x^3 + x}{3x - x^2} = \lim_{x \rightarrow 0} \frac{x(x^2 + 1)}{x(3 - x)} = \lim_{x \rightarrow 0} \frac{x^2 + 1}{3 - x} = \frac{1}{3}$$

$$\begin{aligned}(4) \lim_{x \rightarrow -2} \frac{x^3 + 8}{x^2 - x - 6} &= \lim_{x \rightarrow -2} \frac{(x + 2)(x^2 - 2x + 4)}{(x - 3)(x + 2)} \\ &= \lim_{x \rightarrow -2} \frac{x^2 - 2x + 4}{x - 3} \\ &= -\frac{12}{5}\end{aligned}$$

$$\begin{aligned}(5) \lim_{x \rightarrow 1} \frac{x^2 - 4x + 3}{x^3 - 1} &= \lim_{x \rightarrow 1} \frac{(x - 3)(x - 1)}{(x - 1)(x^2 + x + 1)} \\ &= \lim_{x \rightarrow 1} \frac{x - 3}{x^2 + x + 1} \\ &= -\frac{2}{3}\end{aligned}$$

Limits of Functions 1

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	1	2	3	4

Find the limit value of each of the following functions.

$$(1) \lim_{x \rightarrow +\infty} (2x^2 - 3x + 1) = \lim_{x \rightarrow +\infty} \left[x^2 \left(2 - \frac{3}{x} + \frac{1}{x^2} \right) \right]$$
$$= \infty$$

$$(2) \lim_{x \rightarrow +\infty} (x^3 - 2x^2 + 3x - 4) = \lim_{x \rightarrow +\infty} \left[x^3 \left(1 - \frac{2}{x} + \frac{3}{x^2} - \frac{4}{x^3} \right) \right]$$
$$= \infty$$

$$(3) \lim_{x \rightarrow -\infty} (4x^5 + x^2 + 4)$$

[Sol] Letting $x = -y$, as $x \rightarrow -\infty$, $y \rightarrow \infty$

$$= \lim_{y \rightarrow \infty} (-4y^5 + y^2 + 4)$$
$$= \lim_{y \rightarrow \infty} \left[y^5 \left(-4 + \frac{1}{y^3} + \frac{4}{y^5} \right) \right]$$
$$= -\infty$$

$$(4) \lim_{x \rightarrow -\infty} (2x^4 - x^3 + 2x - 1)$$

[Sol] Letting $x = -y$, as $x \rightarrow -\infty$, $y \rightarrow \infty$

$$= \lim_{y \rightarrow \infty} (2y^4 + y^3 - 2y - 1) = \lim_{y \rightarrow \infty} \left[y^4 \left(2 + \frac{1}{y} - \frac{2}{y^3} - \frac{1}{y^4} \right) \right]$$
$$= \infty$$

$$(5) \lim_{x \rightarrow \infty} \frac{2x+1}{x^2+x-1} = \lim_{x \rightarrow \infty} \frac{\frac{2}{x} + \frac{1}{x^2}}{1 + \frac{1}{x} - \frac{1}{x^2}} = 0$$

$$(6) \lim_{x \rightarrow \infty} \frac{2x^2 - x - 6}{3x^2 - 2x - 8} = \lim_{x \rightarrow \infty} \frac{2 - \frac{1}{x} - \frac{6}{x^2}}{3 - \frac{2}{x} - \frac{8}{x^2}} = \frac{2}{3}$$

$$(7) \lim_{x \rightarrow \infty} \frac{4x^2 - x + 2}{2x - 3} = \lim_{x \rightarrow \infty} \frac{4x - 1 + \frac{2}{x}}{2 - \frac{3}{x}} = \infty$$

$$(8) \lim_{x \rightarrow -\infty} \frac{4x^2 - x + 2}{2x - 3}$$

[Sol] Letting $x = -y$, as $x \rightarrow -\infty$, $y \rightarrow \infty$

$$= \lim_{y \rightarrow \infty} \frac{4y^2 + y + 2}{-2y - 3} = \lim_{y \rightarrow \infty} \frac{4y + 1 + \frac{2}{y}}{-2 - \frac{3}{y}} = -\infty$$

$$(9) \lim_{x \rightarrow \infty} \frac{2x^3 - 7x + 5}{x^3 + 3x^2 - 4} = \lim_{x \rightarrow \infty} \frac{2 - \frac{7}{x^2} + \frac{5}{x^3}}{1 + \frac{3}{x} - \frac{4}{x^3}} = 2$$

Note

In these exercises, we divide the numerator and the denominator by the variable in the denominator with the highest power.

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(mistakes) 0	1	2	3	4

Find the limit value of each of the following functions.

Ex.

$$\lim_{x \rightarrow \infty} \frac{x+2}{\sqrt{x^2+1}+3}$$

[Sol] Dividing the numerator and denominator by x or $\sqrt{x^2}$,

$$\begin{aligned}
 &= \lim_{x \rightarrow \infty} \frac{1 + \frac{2}{x}}{\sqrt{1 + \frac{1}{x^2}} + \frac{3}{x}} \\
 &= 1
 \end{aligned}$$

$$(1) \lim_{x \rightarrow \infty} \frac{3x^2 + \sqrt{x^3+2}}{x^2 - x + 1}$$

[Sol] Dividing the numerator and denominator by x^2 or $\sqrt{x^4}$,

$$\begin{aligned}
 &= \lim_{x \rightarrow \infty} \frac{3 + \sqrt{\frac{1}{x} + \frac{2}{x^4}}}{1 - \frac{1}{x} + \frac{1}{x^2}} = 3
 \end{aligned}$$

$$(2) \lim_{x \rightarrow \infty} \frac{\sqrt{x+1} + \sqrt{3x-1}}{\sqrt{x-1}}$$

[Sol] Dividing the numerator and denominator by $\sqrt{x}$,

$$\begin{aligned}
 &= \lim_{x \rightarrow \infty} \frac{\sqrt{1 + \frac{1}{x}} + \sqrt{3 - \frac{1}{x}}}{\sqrt{1 - \frac{1}{x}}} = 1 + \sqrt{3}
 \end{aligned}$$

$$(3) \lim_{x \rightarrow -\infty} \frac{\sqrt{1+x^2}-1}{x}$$

[Sol] Letting $x = -t$, as $x \rightarrow -\infty$, $t \rightarrow \infty$

$$\begin{aligned}
 &= \lim_{t \rightarrow \infty} \frac{\sqrt{1+t^2}-1}{-t} = \lim_{t \rightarrow \infty} \left(-\sqrt{\frac{1}{t^2} + 1} + \frac{1}{t} \right) = -1
 \end{aligned}$$

$$\begin{aligned}
 (4) \lim_{x \rightarrow 0} \frac{\sqrt{1+x}-1}{x} &= \lim_{x \rightarrow 0} \frac{(\sqrt{1+x}-1)(\sqrt{1+x}+1)}{x(\sqrt{1+x}+1)} \\
 &= \lim_{x \rightarrow 0} \frac{x}{x(\sqrt{1+x}+1)} \\
 &= \lim_{x \rightarrow 0} \frac{1}{\sqrt{1+x}+1} \\
 &= \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 (5) \lim_{x \rightarrow 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} &= \lim_{x \rightarrow 0} \frac{(\sqrt{1+x}-\sqrt{1-x})(\sqrt{1+x}+\sqrt{1-x})}{x(\sqrt{1+x}+\sqrt{1-x})} \\
 &= \lim_{x \rightarrow 0} \frac{2}{\sqrt{1+x}+\sqrt{1-x}} \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 (6) \lim_{x \rightarrow 1} \frac{\sqrt{x+3}-2}{x^2-1} &= \lim_{x \rightarrow 1} \left[\frac{1}{(x+1)(x-1)} \cdot \frac{x-1}{\sqrt{x+3}+2} \right] \\
 &= \lim_{x \rightarrow 1} \frac{1}{(x+1)(\sqrt{x+3}+2)} \\
 &= \frac{1}{8}
 \end{aligned}$$

$$\begin{aligned}
 (7) \lim_{x \rightarrow 2} \frac{\sqrt{x^2+5}-3}{x^2-4} &= \lim_{x \rightarrow 2} \frac{x^2-4}{(x^2-4)(\sqrt{x^2+5}+3)} \\
 &= \lim_{x \rightarrow 2} \frac{1}{\sqrt{x^2+5}+3} \\
 &= \frac{1}{6}
 \end{aligned}$$

Limits of Functions 1

Time : to : Date Name

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(mistakes) 0	-	1	2	3

Find the limit value of each of the following functions.

$$\begin{aligned}
 (1) \lim_{x \rightarrow 0} \frac{x}{\sqrt{x+1}-1} &= \lim_{x \rightarrow 0} \frac{x(\sqrt{x+1}+1)}{(\sqrt{x+1}-1)(\sqrt{x+1}+1)} \\
 &= \lim_{x \rightarrow 0} \frac{x(\sqrt{x+1}+1)}{x} = \lim_{x \rightarrow 0} (\sqrt{x+1}+1) \\
 &= 2
 \end{aligned}$$

$$\begin{aligned}
 (2) \lim_{x \rightarrow 6} \frac{x-6}{\sqrt{x-2}-2} &= \lim_{x \rightarrow 6} \frac{(x-6)(\sqrt{x-2}+2)}{x-6} \\
 &= \lim_{x \rightarrow 6} (\sqrt{x-2}+2) \\
 &= 4
 \end{aligned}$$

$$\begin{aligned}
 (3) \lim_{x \rightarrow 0} \frac{x}{\sqrt{x+5}-\sqrt{5}} &= \lim_{x \rightarrow 0} \frac{x(\sqrt{x+5}+\sqrt{5})}{x} \\
 &= \lim_{x \rightarrow 0} (\sqrt{x+5}+\sqrt{5}) \\
 &= 2\sqrt{5}
 \end{aligned}$$

$$\begin{aligned}
 (4) \lim_{x \rightarrow 0} \frac{\sqrt{1+x+x^2}-1}{\sqrt{1+x}-\sqrt{1-x}} &= \lim_{x \rightarrow 0} \frac{(\sqrt{1+x+x^2}-1)(\sqrt{1+x+x^2}+1)(\sqrt{1+x}+\sqrt{1-x})}{(\sqrt{1+x}-\sqrt{1-x})(\sqrt{1+x}+\sqrt{1-x})(\sqrt{1+x+x^2}+1)} \\
 &= \lim_{x \rightarrow 0} \frac{(1+x)(\sqrt{1+x}+\sqrt{1-x})}{2(\sqrt{1+x+x^2}+1)} \\
 &= \frac{1}{2}
 \end{aligned}$$

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$$\begin{aligned}
 (5) \lim_{x \rightarrow \infty} (\sqrt{x^2 - 4} - x) &= \lim_{x \rightarrow \infty} \frac{(\sqrt{x^2 - 4} - x)(\sqrt{x^2 - 4} + x)}{(\sqrt{x^2 - 4} + x)} \\
 &= \lim_{x \rightarrow \infty} \frac{(x^2 - 4) - x^2}{\sqrt{x^2 - 4} + x} \\
 &= \lim_{x \rightarrow \infty} \frac{-4}{\sqrt{x^2 - 4} + x} \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 (6) \lim_{x \rightarrow \infty} (x - \sqrt{x^2 + 1}) &= \lim_{x \rightarrow \infty} \frac{(x - \sqrt{x^2 + 1})(x + \sqrt{x^2 + 1})}{x + \sqrt{x^2 + 1}} \\
 &= \lim_{x \rightarrow \infty} \frac{-1}{x + \sqrt{x^2 + 1}} \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 (7) \lim_{x \rightarrow \infty} (2x + 1 - \sqrt{4x^2 + 2x + 1}) &= \lim_{x \rightarrow \infty} \frac{(2x + 1 - \sqrt{4x^2 + 2x + 1})(2x + 1 + \sqrt{4x^2 + 2x + 1})}{2x + 1 + \sqrt{4x^2 + 2x + 1}} \\
 &= \lim_{x \rightarrow \infty} \frac{(2x + 1)^2 - (4x^2 + 2x + 1)}{2x + 1 + \sqrt{4x^2 + 2x + 1}} \\
 &= \lim_{x \rightarrow \infty} \frac{2x}{2x + 1 + \sqrt{4x^2 + 2x + 1}} \\
 &= \lim_{x \rightarrow \infty} \frac{2}{2 + \frac{1}{x} + \sqrt{4 + \frac{2}{x} + \frac{1}{x^2}}} \\
 &= \frac{1}{2}
 \end{aligned}$$

Limits of Functions 1

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	1	2	3	4

Find the limit value of each of the following functions.

$$\begin{aligned}
 (1) \quad & \lim_{x \rightarrow 2} \frac{\sqrt{x+2} - \sqrt{3x-2}}{\sqrt{5x-1} - \sqrt{4x+1}} \\
 &= \lim_{x \rightarrow 2} \frac{(\sqrt{x+2} - \sqrt{3x-2})(\sqrt{x+2} + \sqrt{3x-2})(\sqrt{5x-1} + \sqrt{4x+1})}{(\sqrt{5x-1} - \sqrt{4x+1})(\sqrt{5x-1} + \sqrt{4x+1})(\sqrt{x+2} + \sqrt{3x-2})} \\
 &= \lim_{x \rightarrow 2} \frac{(-2x+4)(\sqrt{5x-1} + \sqrt{4x+1})}{(x-2)(\sqrt{x+2} + \sqrt{3x-2})} \\
 &= \lim_{x \rightarrow 2} \frac{-2(\sqrt{5x-1} + \sqrt{4x+1})}{\sqrt{x+2} + \sqrt{3x-2}} \\
 &= -2 \cdot \frac{3+3}{2+2} \\
 &= -3
 \end{aligned}$$

$$(2) \quad \lim_{x \rightarrow \infty} [x(\sqrt{x+1} - \sqrt{x})]$$

$$\begin{aligned}
 &= \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x+1} + \sqrt{x}} \\
 &= \lim_{x \rightarrow \infty} \frac{\sqrt{x}}{\sqrt{1+\frac{1}{x}} + 1} \\
 &= \infty
 \end{aligned}$$

$$(3) \quad \lim_{x \rightarrow -\infty} (x\sqrt{x^2+1} + x^2)$$

[Sol] Letting $x = -t$, as $x \rightarrow -\infty$, $t \rightarrow \infty$

$$\begin{aligned}
 &= \lim_{t \rightarrow \infty} [-t(\sqrt{t^2+1} - t)] \\
 &= \lim_{t \rightarrow \infty} \frac{-t}{\sqrt{t^2+1} + t} \\
 &= \lim_{t \rightarrow \infty} \frac{-1}{\sqrt{1+\frac{1}{t^2}} + 1} = -\frac{1}{2}
 \end{aligned}$$

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$$(4) \lim_{x \rightarrow \infty} (\sqrt{x^2 + 2x + 3} - \sqrt{x^2 - 2x + 3})$$

$$= \lim_{x \rightarrow \infty} \frac{(\sqrt{x^2 + 2x + 3} - \sqrt{x^2 - 2x + 3})(\sqrt{x^2 + 2x + 3} + \sqrt{x^2 - 2x + 3})}{(\sqrt{x^2 + 2x + 3} + \sqrt{x^2 - 2x + 3})}$$

$$= \lim_{x \rightarrow \infty} \frac{(x^2 + 2x + 3) - (x^2 - 2x + 3)}{\sqrt{x^2 + 2x + 3} + \sqrt{x^2 - 2x + 3}}$$

$$= \lim_{x \rightarrow \infty} \frac{4x}{\sqrt{x^2 + 2x + 3} + \sqrt{x^2 - 2x + 3}}$$

$$= \lim_{x \rightarrow \infty} \frac{4}{\sqrt{1 + \frac{2}{x} + \frac{3}{x^2}} + \sqrt{1 - \frac{2}{x} + \frac{3}{x^2}}}$$

$$= \frac{4}{2}$$

$$= 2$$

$$(5) \lim_{x \rightarrow -\infty} (\sqrt{x^2 + 3x - 1} - \sqrt{x^2 - x + 1})$$

[Sol] Letting $x = -t$, as $x \rightarrow -\infty$, $t \rightarrow \infty$

$$= \lim_{t \rightarrow \infty} (\sqrt{t^2 - 3t - 1} - \sqrt{t^2 + t + 1})$$

$$= \lim_{t \rightarrow \infty} \frac{(\sqrt{t^2 - 3t - 1} - \sqrt{t^2 + t + 1})(\sqrt{t^2 - 3t - 1} + \sqrt{t^2 + t + 1})}{(\sqrt{t^2 - 3t - 1} + \sqrt{t^2 + t + 1})}$$

$$= \lim_{t \rightarrow \infty} \frac{-4t - 2}{\sqrt{t^2 - 3t - 1} + \sqrt{t^2 + t + 1}}$$

$$= \lim_{t \rightarrow \infty} \frac{-4 - \frac{2}{t}}{\sqrt{1 - \frac{3}{t} - \frac{1}{t^2}} + \sqrt{1 + \frac{1}{t} + \frac{1}{t^2}}}$$

$$= \frac{-4}{1+1}$$

$$= -2$$

Limits of Functions 1

Time : to : Date Name

100%	90%	80%	70%	60%
(mistake) 0	-	-	1	2

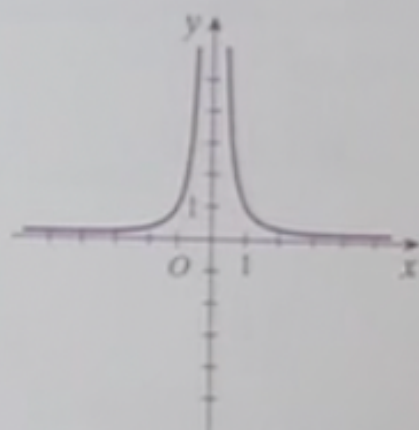
Find the limit value of each of the following functions.

(1) $\lim_{x \rightarrow 0} \frac{1}{x^2}$

[Sol] The graph of $y = \frac{1}{x^2}$ is shown at the right.

When $x = 0$, y has no value, but as $x \rightarrow 0$, $y \rightarrow \boxed{+\infty}$

$$\therefore \lim_{x \rightarrow 0} \frac{1}{x^2} = \boxed{+\infty}$$

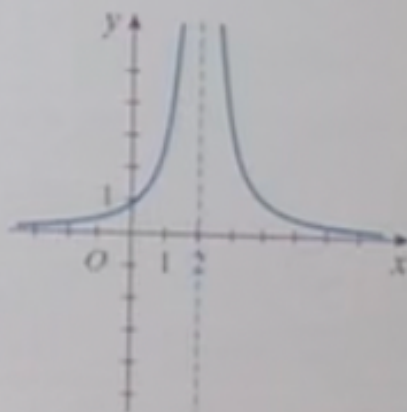


(2) $\lim_{x \rightarrow 2} \frac{3}{(x-2)^2}$

[Sol] The graph of $y = \frac{3}{(x-2)^2}$ is shown at the right.

When $x = 2$, y has no value, but as $x \rightarrow 2$, $y \rightarrow +\infty$

$$\therefore \lim_{x \rightarrow 2} \frac{3}{(x-2)^2} = +\infty$$



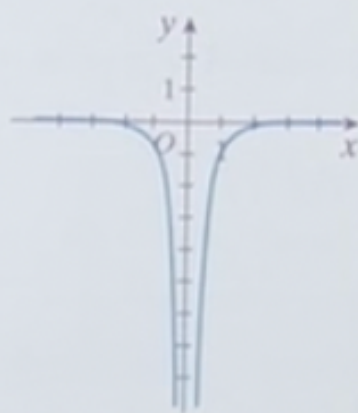
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$$(3) \lim_{x \rightarrow 0} \left(-\frac{1}{x^2} \right)$$

[Sol] The graph of $y = -\frac{1}{x^2}$ is shown at the right.

When $x = 0$, y has no value, but as $x \rightarrow 0$, $y \rightarrow -\infty$

$$\therefore \lim_{x \rightarrow 0} \left(-\frac{1}{x^2} \right) = -\infty$$

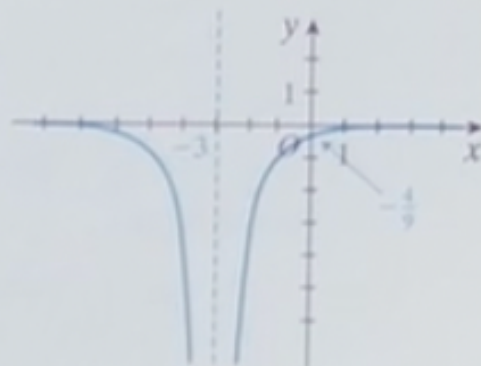


$$(4) \lim_{x \rightarrow -3} \left[-\frac{4}{(x+3)^2} \right]$$

[Sol] The graph of $y = -\frac{4}{(x+3)^2}$ is shown at the right.

When $x = -3$, y has no value, but as $x \rightarrow -3$, $y \rightarrow -\infty$

$$\therefore \lim_{x \rightarrow -3} \left[-\frac{4}{(x+3)^2} \right] = -\infty$$



Limits of Functions 1

Time : to : Date Name

100%	90%	80%	70%	60%~
(mistakes) 0	-	1	-	2-

1. Find the limit value of the following function.

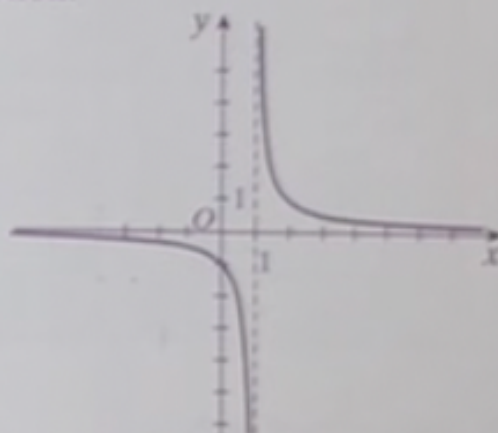
$$\lim_{x \rightarrow 1} \frac{1}{x-1}$$

[Sol] The graph of $y = \frac{1}{x-1}$ is shown at the right.

When $x = 1$, y has no value.

When $x > 1$, as $x \rightarrow 1$, $y \rightarrow +\infty$

When $x < 1$, as $x \rightarrow 1$, $y \rightarrow -\infty$



From the above, when the value of the limit differs depending on the direction of approach to $x = 1$, $\lim_{x \rightarrow 1} \frac{1}{x-1}$ has no value.

Existence of Limits

When considering the function $f(x)$,

- Let us define the "left limit" as the value of the limit when x approaches a from the left. It is expressed as $\lim_{x \rightarrow a^-} f(x)$.
- Let us define the "right limit" as the value of the limit when x approaches a from the right. It is expressed as $\lim_{x \rightarrow a^+} f(x)$.

If $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x)$, then $\lim_{x \rightarrow a} f(x)$ exists.

If $\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$, then $\lim_{x \rightarrow a} f(x)$ does not exist.

In particular, when $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = \alpha$ (where α has a finite value), $\lim_{x \rightarrow a} f(x) = \alpha$, and the value of the limit α exists.

Note

Using 1.,

$\lim_{x \rightarrow 1^-} \frac{1}{x-1}$ is the "left limit" and $\lim_{x \rightarrow 1^+} \frac{1}{x-1}$ is the "right limit."

Since $\lim_{x \rightarrow 1^-} \frac{1}{x-1} \neq \lim_{x \rightarrow 1^+} \frac{1}{x-1}$, $\therefore \lim_{x \rightarrow 1} \frac{1}{x-1}$ does not exist.

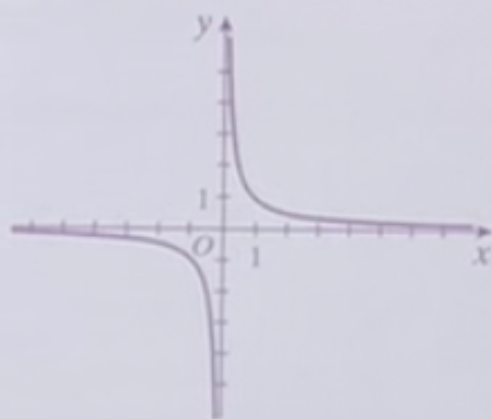
N 137 b

2. Find the limit value of each of the following functions.

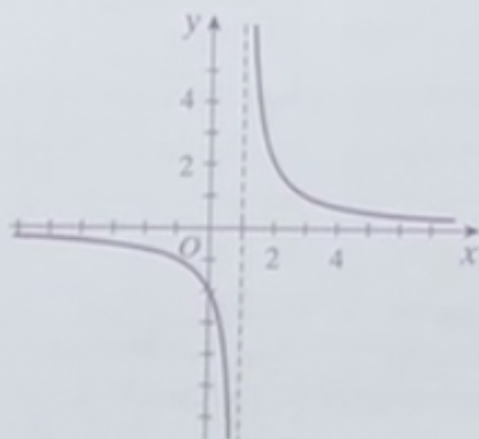
Ex.

$$\lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty$$

$$\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$$

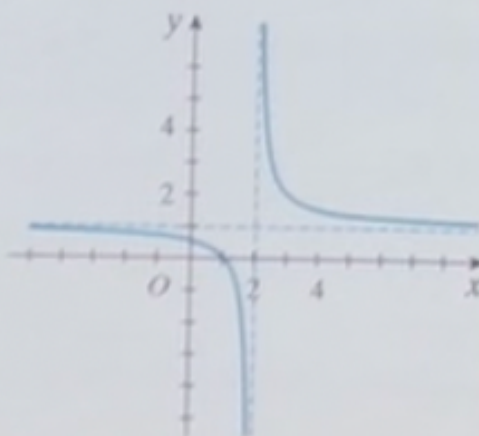


(1) $\lim_{x \rightarrow 1^-} \frac{2}{x-1} = +\infty$



(2) $\lim_{x \rightarrow 1^+} \frac{2}{x-1} = -\infty$

(3) $\lim_{x \rightarrow 2^-} \left(1 + \frac{1}{x-2}\right) = -\infty$



(4) $\lim_{x \rightarrow 2^+} \left(1 + \frac{1}{x-2}\right) = +\infty$

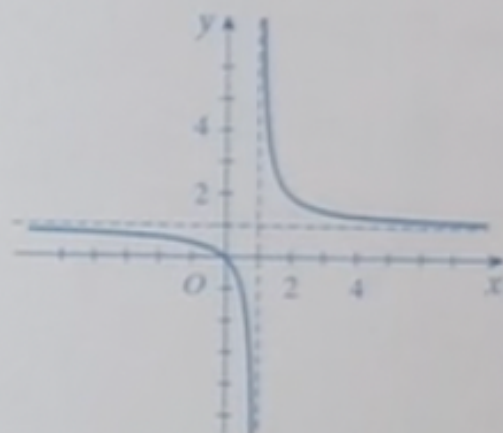
Limits of Functions 1

Time : to : Date Name

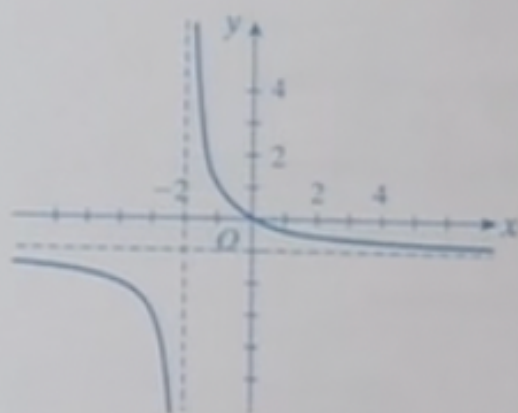
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(mistakes) 0	—	1	2	3

Find the limit value of each of the following functions.

(1) $\lim_{x \rightarrow 1^+} \frac{x}{x-1} = +\infty$



(2) $\lim_{x \rightarrow 1^-} \frac{x}{x-1} = -\infty$

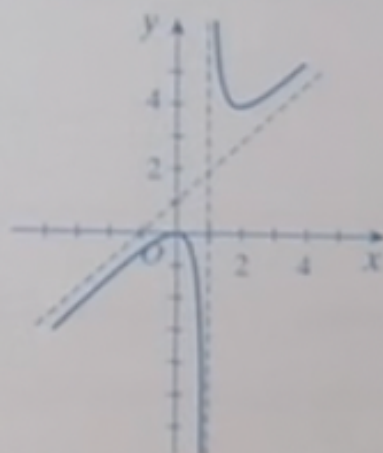


(3) $\lim_{x \rightarrow -2^-} -\frac{x}{x+2} = -\infty$

(4) $\lim_{x \rightarrow -2^+} -\frac{x}{x+2} = +\infty$

(5) $\lim_{x \rightarrow 1^+} \frac{x^2}{x-1} = +\infty$

(6) $\lim_{x \rightarrow 1^-} \frac{x^2}{x-1} = -\infty$



N 138 b

$$(7) \lim_{x \rightarrow 2} \frac{x-1}{(x-2)^2}$$

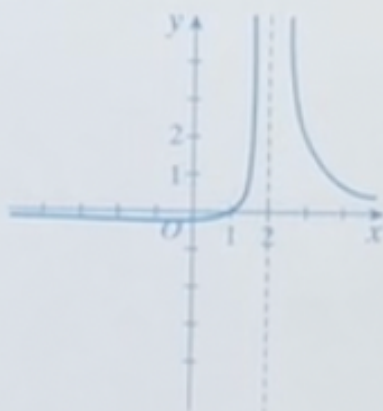
$$[\text{Sol}] \lim_{x \rightarrow 2} (x-1) = \boxed{1}$$

$$\lim_{x \rightarrow 2^+} (x-2)^2 = \boxed{0}$$

$$\lim_{x \rightarrow 2^-} (x-2)^2 = \boxed{0}$$

$$\therefore \lim_{x \rightarrow 2} (x-2)^2 = \boxed{0}$$

$$\therefore \lim_{x \rightarrow 2} \frac{x-1}{(x-2)^2} = \boxed{+\infty}$$



$$(8) \lim_{x \rightarrow 0} \frac{x-2}{x^2-x}$$

$$[\text{Sol}] \lim_{x \rightarrow 0} (x-2) = -2$$

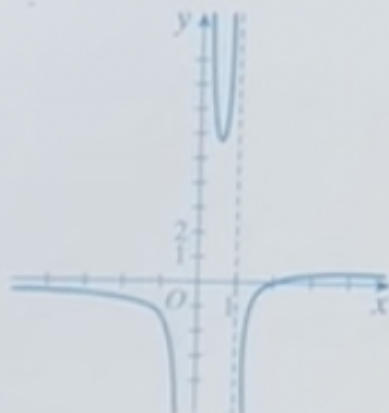
$$\text{When } x \rightarrow 0^+, x^2 - x = x(x-1) \rightarrow 0^-$$

$$\text{When } x \rightarrow 0^-, x^2 - x = x(x-1) \rightarrow 0^+$$

Therefore,

$$\lim_{x \rightarrow 0^+} \frac{x-2}{x^2-x} = +\infty, \quad \lim_{x \rightarrow 0^-} \frac{x-2}{x^2-x} = -\infty$$

Thus, $\lim_{x \rightarrow 0} \frac{x-2}{x^2-x}$ does not exist.



Note

Generally, when $\lim_{x \rightarrow a} f(x) = b$ ($b \neq 0$) and $\lim_{x \rightarrow a} g(x) = 0$, then

$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ has no limit value (no finite value).

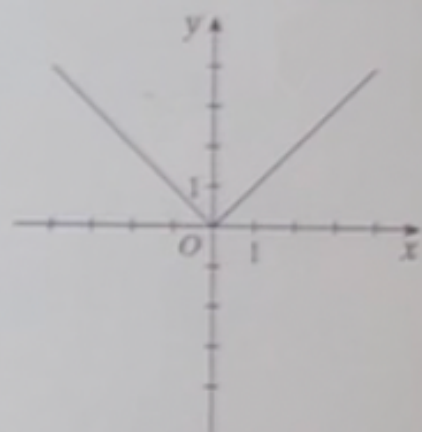
Limits of Functions 1

Time : to : Date Name

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(mistakes) 0	1	2	3	4

1. Find the limit value of each of the following functions.

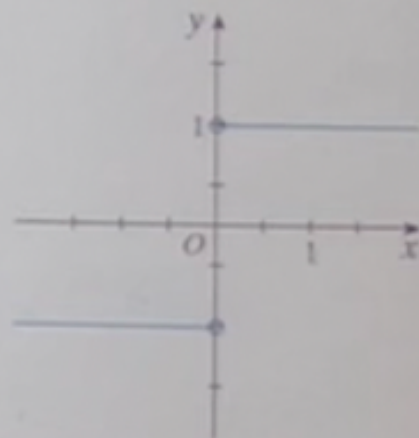
(1) $\lim_{x \rightarrow 0} |x| = 0$



(2) $\lim_{x \rightarrow 0^-} \frac{|x|}{x}$

[Sol] When $x \rightarrow 0^-$, $x < 0$

$\therefore \lim_{x \rightarrow 0^-} \frac{|x|}{x} = -1$



(3) $\lim_{x \rightarrow 0^+} \frac{|x|}{x}$

[Sol] When $x \rightarrow 0^+$, $x > 0$

$\therefore \lim_{x \rightarrow 0^+} \frac{|x|}{x} = 1$

N 139 b

2. Find the limit value of each of the following functions.

$$(1) \lim_{x \rightarrow 0} \frac{|x| - 1}{x}$$

[Sol] When $x > 0$, $|x| = x$

$$\therefore \lim_{x \rightarrow 0^+} \frac{|x| - 1}{x} = \lim_{x \rightarrow 0^+} \frac{x - 1}{x} = \lim_{x \rightarrow 0^+} \left(1 - \frac{1}{x} \right) = -\infty$$

When $x < 0$, $|x| = -x$

$$\therefore \lim_{x \rightarrow 0^-} \frac{|x| - 1}{x} = \lim_{x \rightarrow 0^-} \frac{-x - 1}{x} = \lim_{x \rightarrow 0^-} \left(-1 - \frac{1}{x} \right) = +\infty$$

$$\therefore \lim_{x \rightarrow 0^+} \frac{|x| - 1}{x} \neq \lim_{x \rightarrow 0^-} \frac{|x| - 1}{x}$$

Hence, $\lim_{x \rightarrow 0} \frac{|x| - 1}{x}$ does not exist.

$$(2) \lim_{x \rightarrow 0} \frac{x^2 + x}{|x|}$$

[Sol] When $x > 0$, $\frac{x^2 + x}{|x|} = \frac{x^2 + x}{x} = x + 1$

$$\lim_{x \rightarrow 0^+} \frac{x^2 + x}{|x|} = \lim_{x \rightarrow 0^+} (x + 1) = 1$$

When $x < 0$, $\frac{x^2 + x}{|x|} = \frac{x^2 + x}{-x} = -x - 1$

$$\lim_{x \rightarrow 0^-} \frac{x^2 + x}{|x|} = \lim_{x \rightarrow 0^-} (-x - 1) = -1$$

$$\therefore \lim_{x \rightarrow 0^+} \frac{x^2 + x}{|x|} \neq \lim_{x \rightarrow 0^-} \frac{x^2 + x}{|x|}$$

Thus, $\lim_{x \rightarrow 0} \frac{x^2 + x}{|x|}$ does not exist.

Time : to : Date Name

100%	90%	80%	70%	69%
(mistakes) 0	-	-	-	1-

Find the limit value of each of the following functions.

$$\begin{aligned}
 (1) \quad \lim_{x \rightarrow \infty} \left(x \sqrt{\frac{x+1}{x-1}} - x \right) &= \lim_{x \rightarrow \infty} \left(x \cdot \frac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x-1}} \right) \\
 &= \lim_{x \rightarrow \infty} \left[x \cdot \frac{(\sqrt{x+1} - \sqrt{x-1})(\sqrt{x+1} + \sqrt{x-1})}{\sqrt{x-1}(\sqrt{x+1} + \sqrt{x-1})} \right] \\
 &= \lim_{x \rightarrow \infty} \frac{2x}{(\sqrt{x-1})(\sqrt{x+1} + \sqrt{x-1})} = \lim_{x \rightarrow \infty} \frac{2x}{\sqrt{x^2-1} + (x-1)} \\
 &= \lim_{x \rightarrow \infty} \frac{2}{\sqrt{1 - \frac{1}{x^2}} + 1 - \frac{1}{x}} \\
 &= \frac{2}{1+1} = 1
 \end{aligned}$$

$$(2) \quad \lim_{x \rightarrow a} \frac{\sqrt{\frac{x^2 + ax + a^2}{3}} - a}{x - a} \quad (\text{where } a \text{ is a positive constant})$$

$$\begin{aligned}
 &= \lim_{x \rightarrow a} \frac{\frac{x^2 + ax + a^2}{3} - a^2}{(x - a) \left(\sqrt{\frac{x^2 + ax + a^2}{3}} + a \right)} \\
 &= \lim_{x \rightarrow a} \frac{x^2 + ax - 2a^2}{3(x - a) \left(\sqrt{\frac{x^2 + ax + a^2}{3}} + a \right)} \\
 &= \lim_{x \rightarrow a} \frac{(x + 2a)(x - a)}{3(x - a) \left(\sqrt{\frac{x^2 + ax + a^2}{3}} + a \right)} \\
 &= \lim_{x \rightarrow a} \frac{x + 2a}{3 \left(\sqrt{\frac{x^2 + ax + a^2}{3}} + a \right)} = \frac{a}{\sqrt{a^2 + a} + a} = \frac{a}{a + a} = \frac{1}{2}
 \end{aligned}$$

N 140 b

$$(3) \lim_{x \rightarrow 0} \left(\sqrt{\frac{1}{x^2} + \frac{1}{x}} - \sqrt{\frac{1}{x^2} - \frac{1}{x}} \right)$$

(Analyze the cases $x > 0$ and $x < 0$.)

$$\begin{aligned} [\text{Sol}] \quad \sqrt{\frac{1}{x^2} + \frac{1}{x}} - \sqrt{\frac{1}{x^2} - \frac{1}{x}} &= \sqrt{\frac{1+x}{x^2}} - \sqrt{\frac{1-x}{x^2}} \\ &= \frac{\sqrt{1+x} - \sqrt{1-x}}{|x|} \\ &= \frac{2x}{|x|(\sqrt{1+x} + \sqrt{1-x})} \end{aligned}$$

When $x > 0$,

$$\begin{aligned} &= \lim_{x \rightarrow 0^+} \frac{2x}{x(\sqrt{1+x} + \sqrt{1-x})} = \lim_{x \rightarrow 0^+} \frac{2}{\sqrt{1+x} + \sqrt{1-x}} \\ &= 1 \end{aligned}$$

When $x < 0$,

$$\begin{aligned} &= \lim_{x \rightarrow 0^-} \frac{2x}{-x(\sqrt{1+x} + \sqrt{1-x})} = \lim_{x \rightarrow 0^-} \frac{-2}{\sqrt{1+x} + \sqrt{1-x}} \\ &= -1 \end{aligned}$$

$$\therefore \lim_{x \rightarrow 0^+} \left(\sqrt{\frac{1}{x^2} + \frac{1}{x}} - \sqrt{\frac{1}{x^2} - \frac{1}{x}} \right) \neq \lim_{x \rightarrow 0^-} \left(\sqrt{\frac{1}{x^2} + \frac{1}{x}} - \sqrt{\frac{1}{x^2} - \frac{1}{x}} \right)$$

Thus, $\lim_{x \rightarrow 0} \left(\sqrt{\frac{1}{x^2} + \frac{1}{x}} - \sqrt{\frac{1}{x^2} - \frac{1}{x}} \right)$ does not exist.

Limits of Functions 2

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	-	-	-

Determine the values of the constants a and b that satisfy the following equalities.

$$(1) \lim_{x \rightarrow 2} \frac{x^2 - ax + b}{x - 2} = 3 \quad \dots \textcircled{1}$$

[Sol] Since $\lim_{x \rightarrow 2} (x - 2) = 0$,

The limit of the numerator must also be equal to zero.

$$\therefore \lim_{x \rightarrow 2} (x^2 - ax + b) = 0$$

$$\text{Therefore, } 4 - 2a + b = 0$$

$$\therefore b = \boxed{2a - 4} \quad \dots \textcircled{2}$$

Substituting $\textcircled{2}$ into $\textcircled{1}$,

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{x^2 - ax + \boxed{2a - 4}}{x - 2} &= \lim_{x \rightarrow 2} \frac{(x - 2)(x + 2 - a)}{x - 2} \\ &= \lim_{x \rightarrow 2} (x + 2 - a) \\ &= 4 - a \end{aligned}$$

$$\text{From } \textcircled{1}, \quad 4 - a = \boxed{3}$$

$$\therefore a = \boxed{1}$$

$$\text{From } \textcircled{2}, \quad b = \boxed{-2}$$

$$\text{Answer: } a = \boxed{1}, b = \boxed{-2}$$

N 141 b

$$(2) \lim_{x \rightarrow 2} \frac{2x^2 + ax + b}{x^2 - x - 2} = \frac{5}{3} \quad \dots \textcircled{1}$$

[Sol] Since $\lim_{x \rightarrow 2} (x^2 - x - 2) = 0$,

The limit of the numerator must also be equal to zero.

$$\therefore \lim_{x \rightarrow 2} (2x^2 + ax + b) = 0$$

$$\text{Therefore, } 2a + b + 8 = 0$$

$$\therefore b = -2(a + 4) \quad \dots \textcircled{2}$$

Substituting $\textcircled{2}$ into $\textcircled{1}$,

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{2x^2 + ax - 2(a + 4)}{x^2 - x - 2} &= \lim_{x \rightarrow 2} \frac{(x - 2)(2x + a + 4)}{(x - 2)(x + 1)} \\ &= \lim_{x \rightarrow 2} \frac{2x + a + 4}{x + 1} \\ &= \frac{a + 8}{3} \end{aligned}$$

$$\text{From } \textcircled{1}, \frac{a + 8}{3} = \frac{5}{3}$$

$$\therefore a = -3$$

$$\text{From } \textcircled{2}, \quad b = -2$$

Answer: $a = -3, b = -2$

Note: Given $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = c (c \neq 0)$, when $\lim_{x \rightarrow a} g(x) = 0$, then $\lim_{x \rightarrow a} f(x) = 0$.

Time : to : Date Name

100%	90%	80%	70%	60%~
(mistakes) 0	-	-	-	1-

Determine the values of the constants a and b that satisfy the following equalities.

$$(1) \lim_{x \rightarrow 1} \frac{a\sqrt{x+1} - b}{x-1} = \sqrt{2} \quad \dots \textcircled{1}$$

[Sol] Since $\lim_{x \rightarrow 1} (x-1) = 0$,

The limit of the numerator must also be equal to zero.

$$\therefore \lim_{x \rightarrow 1} (a\sqrt{x+1} - b) = 0$$

Therefore, $a\sqrt{2} - b = 0$

$$\therefore b = a\sqrt{2} \quad \dots \textcircled{2}$$

Substituting $\textcircled{2}$ into $\textcircled{1}$,

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{a\sqrt{x+1} - a\sqrt{2}}{x-1} &= \lim_{x \rightarrow 1} \frac{a(\sqrt{x+1} - \sqrt{2})}{x-1} \\ &= \lim_{x \rightarrow 1} \frac{a(x-1)}{(x-1)(\sqrt{x+1} + \sqrt{2})} \\ &= \lim_{x \rightarrow 1} \frac{a}{\sqrt{x+1} + \sqrt{2}} \\ &= \frac{a}{\sqrt{2} + \sqrt{2}} = \frac{a}{2\sqrt{2}} \\ &= \frac{a\sqrt{2}}{4} \end{aligned}$$

$$\text{From } \textcircled{1}, \frac{a\sqrt{2}}{4} = \sqrt{2}$$

$$\therefore a = 4$$

$$\text{From } \textcircled{2}, b = 4\sqrt{2}$$

Answer: $a = 4, b = 4\sqrt{2}$

N 142 b

$$(2) \lim_{x \rightarrow \infty} \frac{ax^2 + bx + 5}{x + 3} = 3$$

$$[\text{Sol}] \lim_{x \rightarrow \infty} \frac{ax^2 + bx + 5}{x + 3} = \lim_{x \rightarrow \infty} \frac{ax + b + \frac{5}{x}}{1 + \frac{3}{x}} = 3$$

$$\text{Since } \lim_{x \rightarrow \infty} \left(1 + \frac{3}{x}\right) = 1,$$

The limit of the numerator must be equal to 3.

$$\therefore \lim_{x \rightarrow \infty} \left(ax + b + \frac{5}{x}\right) = 3$$

This is true only when $a = 0$ and $b = 3$.

Answer: $a = 0, b = 3$

Limits of Functions 2

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	-	-	1-

Determine the values of the constants a and b that satisfy the following equalities.

$$(1) \lim_{x \rightarrow -1} \frac{\sqrt{x^2 + ax + b}}{x^2 - 1} = \frac{1}{2} \quad \dots \textcircled{1}$$

[Sol] Since $\lim_{x \rightarrow -1} (x^2 - 1) = 0$,

The limit of the numerator must also be equal to zero.

$$\therefore \lim_{x \rightarrow -1} (\sqrt{x^2 + ax + b}) = 0$$

$$\text{Therefore, } \sqrt{1 - a + b} = 0$$

$$\therefore b = -\sqrt{1 - a} \quad \dots \textcircled{2}$$

Substituting $\textcircled{2}$ into $\textcircled{1}$,

$$\begin{aligned} & \lim_{x \rightarrow -1} \frac{\sqrt{x^2 + ax} - \sqrt{1 - a}}{x^2 - 1} \\ &= \lim_{x \rightarrow -1} \frac{(\sqrt{x^2 + ax} - \sqrt{1 - a})(\sqrt{x^2 + ax} + \sqrt{1 - a})}{(x^2 - 1)(\sqrt{x^2 + ax} + \sqrt{1 - a})} \\ &= \lim_{x \rightarrow -1} \frac{x^2 + ax - (1 - a)}{(x^2 - 1)(\sqrt{x^2 + ax} + \sqrt{1 - a})} \\ &= \lim_{x \rightarrow -1} \frac{(x + 1)(x + a - 1)}{(x + 1)(x - 1)(\sqrt{x^2 + ax} + \sqrt{1 - a})} \\ &= \lim_{x \rightarrow -1} \frac{x + a - 1}{(x - 1)(\sqrt{x^2 + ax} + \sqrt{1 - a})} \\ &= \frac{a - 2}{-2(\sqrt{1 - a} + \sqrt{1 - a})} = \frac{a - 2}{-4\sqrt{1 - a}} = \frac{2 - a}{4\sqrt{1 - a}} \end{aligned}$$

$$\text{From } \textcircled{1}, \frac{2 - a}{4\sqrt{1 - a}} = \frac{1}{2}$$

$$\therefore a = 0$$

$$\text{From } \textcircled{2}, b = -1$$

Answer: $a = 0, b = -1$

N 143 b

$$(2) \lim_{x \rightarrow 1} \frac{(a+1)x+b}{\sqrt{3x+1}-\sqrt{x+3}} = 4 \quad \dots \textcircled{1}$$

[Sol] Since $\lim_{x \rightarrow 1} (\sqrt{3x+1}-\sqrt{x+3}) = 0$,

The limit of the numerator must also be equal to zero.

$$\therefore \lim_{x \rightarrow 1} [(a+1)x+b] = 0$$

Therefore, $(a+1+b) = 0$

$$\therefore b = -a-1 \quad \dots \textcircled{2}$$

Substituting $\textcircled{2}$ into $\textcircled{1}$,

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{(a+1)x - (a+1)}{\sqrt{3x+1}-\sqrt{x+3}} &= \lim_{x \rightarrow 1} \frac{(x-1)(a+1)}{\sqrt{3x+1}-\sqrt{x+3}} \\ &= \lim_{x \rightarrow 1} \frac{(x-1)(a+1)(\sqrt{3x+1}+\sqrt{x+3})}{(\sqrt{3x+1}-\sqrt{x+3})(\sqrt{3x+1}+\sqrt{x+3})} \\ &= \lim_{x \rightarrow 1} \frac{(x-1)(a+1)(\sqrt{3x+1}+\sqrt{x+3})}{(3x+1)-(x+3)} \\ &= \lim_{x \rightarrow 1} \frac{(a+1)(\sqrt{3x+1}+\sqrt{x+3})}{2} \\ &= \frac{4(a+1)}{2} \\ &= 2a+2 \end{aligned}$$

From $\textcircled{1}$, $2a+2 = 4$

$$\therefore a = 1$$

From $\textcircled{2}$, $b = -2$

Answer: $a = 1, b = -2$

Limits of Functions 2

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	—	—	—	1—

1. Given the function $f(x) = \frac{ax^3 + bx^2 + cx + d}{x^2 + x - 2}$, determine the values of the constants a , b , c and d that satisfy the following equalities.

$$\lim_{x \rightarrow \infty} f(x) = 1 \quad \dots \textcircled{1} \qquad \lim_{x \rightarrow 1} f(x) = 0 \quad \dots \textcircled{2}$$

[Sol] Since $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{ax + b + \frac{c}{x} + \frac{d}{x^2}}{1 + \frac{1}{x} - \frac{2}{x^2}} = 1$,

$$\therefore a = \boxed{0}, b = \boxed{1} \dots \textcircled{3}$$

This condition is necessary for when we substitute ∞ into x .

Also, as $x \rightarrow 1$, the denominator $\rightarrow 0$.

Therefore, the numerator must also $\rightarrow 0$.

Thus, $\lim_{x \rightarrow 1} (x^2 + cx + d) = 1 + c + d = 0$. From substituting the values of a and b into the numerator of the original function.

$$\therefore d = -(c + 1) \quad \dots \textcircled{4}$$

Substituting $\textcircled{3}$ and $\textcircled{4}$ into the original equation,

$$\begin{aligned} \lim_{x \rightarrow 1} f(x) &= \lim_{x \rightarrow 1} \frac{x^2 + cx - (c + 1)}{x^2 + x - 2} \\ &= \lim_{x \rightarrow 1} \frac{(x + c + 1)(x - 1)}{(x + 2)(x - 1)} \\ &= \lim_{x \rightarrow 1} \frac{x + c + 1}{x + 2} \\ &= \frac{c + 2}{3} \end{aligned}$$

$$\text{From } \textcircled{2}, \frac{c + 2}{3} = 0$$

$$\therefore c = -2$$

$$\text{From } \textcircled{4}, d = 1$$

Answer: $a = 0, b = 1, c = -2, d = 1$

N 144 b

2. Given the function $f(x) = \frac{ax^3 - bx^2 + cx - d}{2x^2 - 5x + 3}$, determine the values of the constants a, b, c and d that satisfy the following equalities.

$$\lim_{x \rightarrow \infty} f(x) = 2 \quad \dots \textcircled{1} \quad \lim_{x \rightarrow 1} f(x) = 0 \quad \dots \textcircled{2}$$

[Sol] Since $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{ax - b + \frac{c}{x} - \frac{d}{x^2}}{2 - \frac{5}{x} + \frac{3}{x^2}} = 2,$

$$\therefore a = 0, b = -4 \dots \textcircled{3}$$

Also, as $x \rightarrow 1$, the denominator $\rightarrow 0$.

Therefore, the numerator must also $\rightarrow 0$.

$$\text{Thus, } \lim_{x \rightarrow 1} (4x^2 + cx - d) = 4 + c - d = 0$$

$$\therefore d = 4 + c \quad \dots \textcircled{4}$$

Substituting $\textcircled{3}$ and $\textcircled{4}$ into the original equation,

$$\begin{aligned} \lim_{x \rightarrow 1} f(x) &= \lim_{x \rightarrow 1} \frac{4x^2 + cx - (4 + c)}{2x^2 - 5x + 3} \\ &= \lim_{x \rightarrow 1} \frac{(4x + c + 4)(x - 1)}{(2x - 3)(x - 1)} \\ &= \lim_{x \rightarrow 1} \frac{4x + c + 4}{2x - 3} \\ &= \frac{c + 8}{-1} \end{aligned}$$

$$\begin{aligned} \text{From } \textcircled{2}, \frac{c + 8}{-1} &= 0 \\ \therefore c &= -8 \end{aligned}$$

$$\text{From } \textcircled{4}, \quad d = -4$$

Answer: $a = 0, b = -4, c = -8, d = -4$

Limits of Functions 2

Time : to : Date Name

100%	90%	80%	70%	69%
(mistakes) 0	-	-	-	-

1. Determine the values of the constants a and b that satisfy the following equality.

$$\lim_{x \rightarrow \infty} (\sqrt{4x^2 - 12x + 11} - ax - b) = 0 \quad \dots \textcircled{1}$$

(Hint) First find the condition for a .

[Sol] Letting $f(x) = \sqrt{4x^2 - 12x + 11} - ax - b$,

$$\text{When } a \leq 0, \quad \lim_{x \rightarrow \infty} f(x) = +\infty$$

$$\text{Since } \lim_{x \rightarrow \infty} f(x) = 0, \quad \therefore a > 0 \quad \dots \textcircled{2}$$

Since $x \rightarrow \infty$, this means that $x > 0$.

$$\begin{aligned} \text{Therefore, } f(x) &= \frac{4x^2 - 12x + 11 - (ax + b)^2}{\sqrt{4x^2 - 12x + 11} + ax + b} \\ &= \frac{(4 - a^2)x - (12 + 2ab) + \frac{11 - b^2}{x}}{\sqrt{4 - \frac{12}{x} + \frac{11}{x^2}} + a + \frac{b}{x}} \end{aligned}$$

As $x \rightarrow \infty$, the denominator $\rightarrow 2 + a$.

In order to satisfy $\textcircled{1}$,

The numerator must $\rightarrow 0$.

$$\therefore (4 - a^2) = 0 \quad \dots \textcircled{3} \quad \text{and} \quad (12 + 2ab) = 0 \quad \dots \textcircled{4}$$

From $\textcircled{2}$, $\textcircled{3}$ and $\textcircled{4}$,

$$a = 2, b = -3$$

Answer: $a = 2, b = -3$

N 145 b

2. Determine the values of the constants a and b that satisfy the following equality.

$$\lim_{x \rightarrow \infty} (\sqrt{9x^2 - 3x + 2} - ax + b) = 0 \quad \dots \textcircled{1}$$

[Sol] Letting $f(x) = \sqrt{9x^2 - 3x + 2} - ax + b$,

When $a \leq 0$, $\lim_{x \rightarrow \infty} f(x) = +\infty$

Since $\lim_{x \rightarrow \infty} f(x) = 0$, $\therefore a > 0 \quad \dots \textcircled{2}$

Since $x \rightarrow \infty$, this means that $x > 0$.

$$\begin{aligned} \text{Therefore, } f(x) &= \frac{9x^2 - 3x + 2 - (ax - b)^2}{\sqrt{9x^2 - 3x + 2} + ax - b} \\ &= \frac{(9 - a^2)x - (3 - 2ab) + \frac{2 - b^2}{x}}{\sqrt{9 - \frac{3}{x} + \frac{2}{x^2}} + a - \frac{b}{x}} \end{aligned}$$

As $x \rightarrow \infty$, the denominator $\rightarrow (3 + a)$.

In order to satisfy $\textcircled{1}$,

The numerator must $\rightarrow 0$.

$$\therefore (9 - a^2) = 0 \quad \dots \textcircled{3} \quad \text{and} \quad (3 - 2ab) = 0 \quad \dots \textcircled{4}$$

From $\textcircled{2}$, $\textcircled{3}$ and $\textcircled{4}$,

$$a = 3, b = \frac{1}{2}$$

Answer: $a = 3, b = \frac{1}{2}$

Limits of Functions 2

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	1	2	3	4

Find the limit values of the following functions.

$$(1) \lim_{x \rightarrow -\infty} \left(\frac{4}{5}\right)^x = 0$$

$$(2) \lim_{x \rightarrow -\infty} \left(\frac{9}{10}\right)^x = \infty$$

$$(3) \lim_{x \rightarrow \infty} \left(\frac{9}{8}\right)^x = \infty$$

$$(4) \lim_{x \rightarrow -\infty} \left(\frac{5}{4}\right)^x = 0$$

$$(5) \lim_{x \rightarrow \infty} \frac{2^{x-1}}{1+2^x} = \lim_{x \rightarrow \infty} \frac{\frac{2^x}{2}}{1+2^x} = \lim_{x \rightarrow \infty} \frac{2^x}{2(1+2^x)} = \frac{1}{2}$$

$$(6) \lim_{x \rightarrow \infty} \frac{\left(\frac{1}{2}\right)^{x-1}}{1+\left(\frac{1}{2}\right)^x} = \frac{0}{1+0} = 0$$

Note: $\lim_{x \rightarrow -\infty} a^x = \lim_{x \rightarrow \infty} \left(\frac{1}{a}\right)^x$

N 146 b

$$(7) \lim_{x \rightarrow \infty} a^x \quad (a > 0)$$

[Sol] (a) When $a > 1$,

$$\lim_{x \rightarrow \infty} a^x = \boxed{\infty}$$

(b) When $a = 1$,

$$\lim_{x \rightarrow \infty} a^x = \boxed{1}$$

(c) When $0 < a < 1$,

$$\lim_{x \rightarrow \infty} a^x = \boxed{0}$$

$$(8) \lim_{x \rightarrow \infty} \frac{a^{x-1}}{1+a^x} \quad (a > 0)$$

[Sol]

(a) When $a > 1$,

Dividing both the numerator and the denominator by a^x ,

$$\frac{a^{x-1}}{1+a^x} = \frac{a^{-1}}{a^{-x}+1}$$

Since $\lim_{x \rightarrow \infty} a^{-x} = 0$,

$$\lim_{x \rightarrow \infty} \frac{a^{x-1}}{1+a^x} = \lim_{x \rightarrow \infty} \frac{a^{-1}}{a^{-x}+1} = \frac{a^{-1}}{0+1} = \frac{1}{a}$$

(b) When $0 < a < 1$,

Since $\lim_{x \rightarrow \infty} a^x = 0$, $\lim_{x \rightarrow \infty} a^{x-1} = 0$

$$\lim_{x \rightarrow \infty} \frac{a^{x-1}}{1+a^x} = \frac{0}{1+0} = 0$$

(c) When $a = 1$,

Since $\lim_{x \rightarrow \infty} a^x = 1$, $\lim_{x \rightarrow \infty} a^{x-1} = 1$

$$\lim_{x \rightarrow \infty} \frac{a^{x-1}}{1+a^x} = \frac{1}{1+1} = \frac{1}{2}$$

Limits of Functions 2

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	1	2	3	4

Find the limit values of the following functions.

$$(1) \lim_{x \rightarrow \infty} \frac{3^x - 3^{-x}}{3^x + 3^{-x}} = \lim_{x \rightarrow \infty} \frac{1 - 3^{-2x}}{1 + 3^{-2x}} = 1$$

$$(2) \lim_{x \rightarrow \infty} \frac{\left(\frac{2}{3}\right)^x - \left(\frac{2}{3}\right)^{-x}}{\left(\frac{2}{3}\right)^x + \left(\frac{2}{3}\right)^{-x}} = \lim_{x \rightarrow \infty} \frac{\left(\frac{2}{3}\right)^x - \frac{1}{\left(\frac{2}{3}\right)^x}}{\left(\frac{2}{3}\right)^x + \frac{1}{\left(\frac{2}{3}\right)^x}} = \lim_{x \rightarrow \infty} \frac{\left(\frac{2}{3}\right)^{2x} - 1}{\left(\frac{2}{3}\right)^{2x} + 1}$$

$$= -1$$

$$(3) \lim_{x \rightarrow 0^+} \frac{2^{\frac{1}{x}}}{1 + 2^{\frac{1}{x}}} = \lim_{x \rightarrow 0^+} \frac{1}{2^{-\frac{1}{x}} + 1}$$

[Sol] As $x \rightarrow 0^+$, $-\frac{1}{x} \rightarrow -\infty$

$$\therefore \lim_{x \rightarrow 0^+} 2^{-\frac{1}{x}} = 0$$

Therefore, $\lim_{x \rightarrow 0^+} \frac{2^{\frac{1}{x}}}{1 + 2^{\frac{1}{x}}} = \lim_{x \rightarrow 0^+} \frac{1}{2^{-\frac{1}{x}} + 1} = 1$

$$(4) \lim_{x \rightarrow 0^-} \frac{2^{\frac{1}{x}}}{1 + 2^{\frac{1}{x}}} = \lim_{x \rightarrow 0^-} \frac{1}{2^{-\frac{1}{x}} + 1}$$

[Sol] As $x \rightarrow 0^-$, $-\frac{1}{x} \rightarrow +\infty$

$$\therefore \lim_{x \rightarrow 0^-} 2^{-\frac{1}{x}} = \infty$$

Therefore, $\lim_{x \rightarrow 0^-} \frac{2^{\frac{1}{x}}}{1 + 2^{\frac{1}{x}}} = \lim_{x \rightarrow 0^-} \frac{1}{2^{-\frac{1}{x}} + 1} = 0$

N 147 b

$$(5) \lim_{x \rightarrow 0} \frac{1}{1 + 2^{\frac{1}{x}}}$$

[Sol] As $x \rightarrow 0$, $\frac{1}{x^2} \rightarrow \boxed{+\infty}$
 $\therefore \lim_{x \rightarrow 0} 2^{\frac{1}{x}} = +\infty$

Therefore, $\lim_{x \rightarrow 0} \frac{1}{1 + 2^{\frac{1}{x}}} = 0$

$$(6) \lim_{x \rightarrow \infty} \frac{1 - 2^{-x^2}}{1 + 2^{\frac{1}{x}}}$$

[Sol] As $x \rightarrow \infty$, $-x^2 \rightarrow -\infty$, $\frac{1}{x^2} \rightarrow 0$,

$\therefore \lim_{x \rightarrow \infty} 2^{-x^2} = 0$, $\lim_{x \rightarrow \infty} 2^{\frac{1}{x}} = 1$

Therefore, $\lim_{x \rightarrow \infty} \frac{1 - 2^{-x^2}}{1 + 2^{\frac{1}{x}}} = \frac{1 - 0}{1 + 1} = \frac{1}{2}$

$$(7) \lim_{x \rightarrow \infty} \frac{a^x + 1}{a^x + a} \quad (a > 0)$$

[Sol] (a) When $0 < a < 1$,

$$\lim_{x \rightarrow \infty} a^x = 0$$

$$\therefore \lim_{x \rightarrow \infty} \frac{a^x + 1}{a^x + a} = \frac{0 + 1}{0 + a} = \frac{1}{a}$$

(b) When $a = 1$,

$$\lim_{x \rightarrow \infty} a^x = 1$$

$$\therefore \lim_{x \rightarrow \infty} \frac{a^x + 1}{a^x + a} = \frac{1 + 1}{1 + 1} = 1$$

(c) When $a > 1$,

$$\lim_{x \rightarrow \infty} a^{-x} = 0$$

$$\therefore \lim_{x \rightarrow \infty} \frac{a^x + 1}{a^x + a} = \lim_{x \rightarrow \infty} \frac{1 + a^{-x}}{1 + a^{-x} \cdot a} = \frac{1 + 0}{1 + 0 \cdot a} = 1$$

Limits of Functions 2

Time : : to : : Date : Name :

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Find the limit values of the following functions.

Ex.

$$\begin{aligned}\lim_{x \rightarrow +\infty} [\log(x+1) - \log x] &= \lim_{x \rightarrow +\infty} \left(\log \frac{x+1}{x} \right) \\ &= \lim_{x \rightarrow +\infty} \left[\log \left(1 + \frac{1}{x} \right) \right] \\ &= \log 1 \\ &= 0\end{aligned}$$

$$\begin{aligned}(1) \lim_{x \rightarrow +\infty} [\log x - \log(x-1)] &= \lim_{x \rightarrow +\infty} \left(\log \frac{x}{x-1} \right) \\ &= \lim_{x \rightarrow +\infty} \left(\log \frac{1}{1 - \frac{1}{x}} \right) \\ &= \log 1 = 0\end{aligned}$$

$$\begin{aligned}(2) \lim_{x \rightarrow 2} (\log|x^2 - 4| - \log|x^3 - 8|) &= \lim_{x \rightarrow 2} \left(\log \left| \frac{x^2 - 4}{x^3 - 8} \right| \right) \\ &= \lim_{x \rightarrow 2} \left[\log \left| \frac{(x-2)(x+2)}{(x-2)(x^2 + 2x + 4)} \right| \right] \\ &= \lim_{x \rightarrow 2} \left(\log \left| \frac{x+2}{x^2 + 2x + 4} \right| \right) \\ &= \log \frac{1}{3} = -\log 3\end{aligned}$$

$$\begin{aligned}(3) \lim_{x \rightarrow 1} (\log|x^2 - 3x + 2| - \log|x^2 - 4x + 3|) \\ &= \lim_{x \rightarrow 1} \left(\log \left| \frac{x^2 - 3x + 2}{x^2 - 4x + 3} \right| \right) \\ &= \lim_{x \rightarrow 1} \left[\log \left| \frac{(x-2)(x-1)}{(x-3)(x-1)} \right| \right] \\ &= \lim_{x \rightarrow 1} \left(\log \left| \frac{x-2}{x-3} \right| \right) = \log \frac{1}{2} = -\log 2\end{aligned}$$

N 148 b

$$\begin{aligned}
 (4) \lim_{x \rightarrow \infty} [2 \log(x-1) - \log(x^2 - 3x + 3)] &= \lim_{x \rightarrow \infty} \left[\log \frac{(x-1)^2}{x^2 - 3x + 3} \right] \\
 &= \lim_{x \rightarrow \infty} \left(\log \frac{x^2 - 2x + 1}{x^2 - 3x + 3} \right) \\
 &= \lim_{x \rightarrow \infty} \left(\log \frac{1 - \frac{2}{x} + \frac{1}{x^2}}{1 - \frac{3}{x} + \frac{3}{x^2}} \right) \\
 &= \log 1 \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 (5) \lim_{x \rightarrow \infty} [\log(x - \sqrt{x^2 - 1})] &= \lim_{x \rightarrow \infty} \left[\log \frac{x^2 - (x^2 - 1)}{x + \sqrt{x^2 - 1}} \right] \\
 &= \lim_{x \rightarrow \infty} \left(\log \frac{1}{x + \sqrt{x^2 - 1}} \right)
 \end{aligned}$$

$$\text{As } x \rightarrow \infty, \quad \frac{1}{x + \sqrt{x^2 - 1}} \rightarrow 0^+$$

$$\therefore \lim_{x \rightarrow \infty} [\log(x - \sqrt{x^2 - 1})] = -\infty$$

$$\begin{aligned}
 (6) \lim_{x \rightarrow \infty} [\log(2x + 1 - \sqrt{4x^2 + 2x + 1})] \\
 &= \lim_{x \rightarrow \infty} \left[\log \frac{(2x + 1)^2 - (4x^2 + 2x + 1)}{2x + 1 + \sqrt{4x^2 + 2x + 1}} \right] \\
 &= \lim_{x \rightarrow \infty} \left(\log \frac{2x}{2x + 1 + \sqrt{4x^2 + 2x + 1}} \right) \\
 &= \lim_{x \rightarrow \infty} \left(\log \frac{2}{2 + \frac{1}{x} + \sqrt{4 + \frac{2}{x} + \frac{1}{x^2}}} \right) \\
 &= \log \frac{2}{2 + \sqrt{4}} \\
 &= -\log 2
 \end{aligned}$$

Limits of Functions 2

Time : to : Date Name

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(mistakes) 0	-	1	-	2-

Given that $\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e$, find the following limit values.

(1) $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x$

[Sol] Letting $\frac{1}{x} = h$, as $x \rightarrow \infty$, $h \rightarrow \boxed{0}$.

$$\therefore \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = \lim_{h \rightarrow 0} (1+h)^{\frac{1}{h}} = \boxed{e}$$

(2) $\lim_{x \rightarrow 0} (1-x)^{\frac{1}{x}}$

[Sol] Letting $-x = h$, as $x \rightarrow 0$, $h \rightarrow 0$.

$$\therefore \lim_{x \rightarrow 0} (1-x)^{\frac{1}{x}} = \lim_{h \rightarrow 0} (1+h)^{-\frac{1}{h}}$$

$$= \lim_{h \rightarrow 0} \frac{1}{(1+h)^{\frac{1}{h}}}$$

$$= \frac{1}{e}$$

(3) $\lim_{x \rightarrow 0} (1+3x)^{\frac{1}{x}}$

[Sol] Letting $3x = h$, as $x \rightarrow 0$, $h \rightarrow 0$.

$$\therefore \lim_{x \rightarrow 0} (1+3x)^{\frac{1}{x}} = \lim_{h \rightarrow 0} (1+h)^{\frac{1}{h/3}}$$

$$= \lim_{h \rightarrow 0} [(1+h)^{\frac{1}{h}}]^3$$

$$= e^3$$

N 149 b

$$(4) \lim_{x \rightarrow \infty} \left(1 + \frac{a}{x}\right)^x$$

[Sol] Letting $\frac{a}{x} = h$, as $x \rightarrow \infty$, $h \rightarrow 0$.

$$\begin{aligned}\therefore \lim_{x \rightarrow \infty} \left(1 + \frac{a}{x}\right)^x &= \lim_{h \rightarrow 0} (1 + h)^{\frac{1}{h}} \\ &= \lim_{h \rightarrow 0} [(1 + h)^{\frac{1}{h}}]^a \\ &= e^a\end{aligned}$$

$$(5) \lim_{x \rightarrow 0} \frac{e^x - 1}{x}$$

(Hint) Let $e^x - 1 = h$

[Sol] Letting $e^x - 1 = h$,

$$e^x = 1 + h$$

$$\therefore x = \log_e(1 + h)$$

However, as $x \rightarrow 0$, $h \rightarrow 0$.

$$\begin{aligned}\text{Therefore, } \lim_{x \rightarrow 0} \frac{e^x - 1}{x} &= \lim_{h \rightarrow 0} \frac{h}{\log_e(1 + h)} \\ &= \lim_{h \rightarrow 0} \frac{1}{\frac{1}{h} \log_e(1 + h)} \\ &= \lim_{h \rightarrow 0} \frac{1}{\log_e(1 + h)^{\frac{1}{h}}} \\ &= \frac{1}{\log_e e} \\ &= 1\end{aligned}$$

Limits of Functions 2

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	-	-	-

1. Find the limit values of the following function.

$$\lim_{x \rightarrow \infty} \frac{a^x - a^{-x}}{a^x + a^{-x}} \quad (a > 0)$$

[Sol]

(a) When $0 < a < 1$, $\lim_{x \rightarrow \infty} a^x = 0$

$$\therefore \lim_{x \rightarrow \infty} \frac{a^x - a^{-x}}{a^x + a^{-x}} = \lim_{x \rightarrow \infty} \frac{a^{2x} - 1}{a^{2x} + 1} = \frac{0 - 1}{0 + 1} = -1$$

(b) When $a = 1$, $\lim_{x \rightarrow \infty} a^x = 1$

$$\therefore \lim_{x \rightarrow \infty} \frac{a^x - a^{-x}}{a^x + a^{-x}} = \lim_{x \rightarrow \infty} \frac{a^{2x} - 1}{a^{2x} + 1} = \frac{1 - 1}{1 + 1} = 0$$

(c) When $a > 1$, $\lim_{x \rightarrow \infty} a^x = +\infty$

$$\therefore \lim_{x \rightarrow \infty} \frac{a^x - a^{-x}}{a^x + a^{-x}} = \lim_{x \rightarrow \infty} \frac{a^{2x} - 1}{a^{2x} + 1} = \lim_{x \rightarrow \infty} \frac{1 - \frac{1}{a^{2x}}}{1 + \frac{1}{a^{2x}}} = \frac{1 - 0}{1 + 0} = 1$$

2. Find the limit value of the following function.

$$\lim_{x \rightarrow 0} (2\log|\sqrt{1+x}-1| - \log|\sqrt{1+x^2}-1|)$$

$$= \lim_{x \rightarrow 0} \left[\log \left| \frac{(\sqrt{1+x}-1)^2}{\sqrt{1+x^2}-1} \right| \right]$$

$$= \lim_{x \rightarrow 0} \left[\log \left| \frac{(\sqrt{1+x}-1)^2(\sqrt{1+x}+1)^2(\sqrt{1+x^2}+1)}{(\sqrt{1+x^2}-1)(\sqrt{1+x^2}+1)(\sqrt{1+x}+1)^2} \right| \right]$$

$$= \lim_{x \rightarrow 0} \left[\log \frac{(x)^2(\sqrt{1+x^2}+1)}{x^2(\sqrt{1+x}+1)^2} \right]$$

$$= \lim_{x \rightarrow 0} \left[\log \frac{\sqrt{1+x^2}+1}{(\sqrt{1+x}+1)^2} \right]$$

$$= \log \frac{2}{2^2}$$

$$= -\log 2$$

N 150 b

3. Determine the values of the constants a and b that satisfy the following equality.

$$\lim_{x \rightarrow 1} \frac{x^2 + ax + b}{x^2 + 2x - 3} = \frac{7}{4} \quad \dots \textcircled{1}$$

[Sol] Since $\lim_{x \rightarrow 1} (x^2 + 2x - 3) = 0$,

The limit of the numerator must also be equal to zero.

$$\therefore \lim_{x \rightarrow 1} (x^2 + ax + b) = 0$$

Therefore, $1 + a + b = 0$

$$\therefore b = -a - 1 \quad \dots \textcircled{2}$$

Substituting $\textcircled{2}$ into $\textcircled{1}$,

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{x^2 + ax + (-a - 1)}{x^2 + 2x - 3} &= \lim_{x \rightarrow 1} \frac{(x - 1)(x + a + 1)}{(x - 1)(x + 3)} \\ &= \lim_{x \rightarrow 1} \frac{x + a + 1}{x + 3} \\ &= \frac{a + 2}{4} \end{aligned}$$

From $\textcircled{1}$ and $\textcircled{2}$, $a = 5$ and $b = -6$

Answer: $a = 5, b = -6$

Limits of Trigonometric Functions

Time : to : Date Name

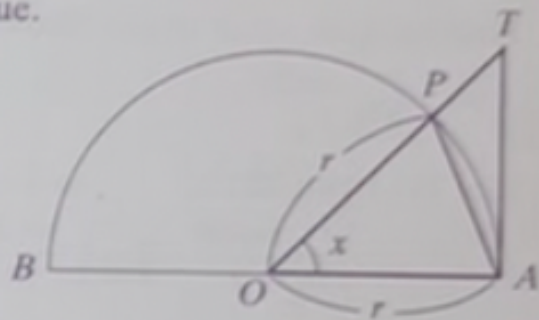
100%	90%	80%	70%	60%~
(mistakes) 0	-	-	-	1-

1. Prove that the following equality is true.

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

[Sol] In the diagram on the right,

Let $0 < x < \frac{\pi}{2}$, and let AT be the tangent line to the semicircle at A .



Given that the area of $\triangle OAP = \frac{1}{2}r^2 \sin x$,

Comparing the areas of $\triangle OAP$, sector OAP and $\triangle OAT$:

$$\triangle OAP < \text{sector } OAP < \triangle OAT$$

Therefore,

$$\frac{1}{2}r^2 \sin x < \frac{1}{2}r^2 x < \frac{1}{2}r \cdot AT \quad \therefore \frac{1}{2}r^2 \sin x < \frac{1}{2}r^2 x < \frac{1}{2}r \cdot r \tan x$$

$$\text{Thus, } \sin x < x < \tan x$$

Since $\sin x > 0$, we can divide all terms by $\sin x$:

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

Since all the terms are positive, taking their reciprocals,

$$1 > \frac{\sin x}{x} > \cos x$$

$$\text{As } x \rightarrow 0^+, \cos x \rightarrow 1, \quad \text{Therefore as } x \rightarrow 0^+, \frac{\sin x}{x} \rightarrow 1.$$

As $x \rightarrow 0^-$, $x < 0$.

Thus, we let $x = -y$.

As $x \rightarrow 0^-$, $y \rightarrow 0^+$

$$\frac{\sin x}{x} = \frac{\sin(-y)}{-y} = \frac{\sin y}{y} \rightarrow 1$$

Since the limit value is 1, regardless of the direction from which x approaches 0,

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

N 151 b

Formula

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

2. Find the limit value of the following function.

$$\lim_{\theta \rightarrow 0} \frac{\sin 2\theta}{\theta}$$

[Sol 1] Letting $2\theta = t$, as $\theta \rightarrow 0$, $t \rightarrow \boxed{0}$.

$$\begin{aligned} \therefore \lim_{\theta \rightarrow 0} \frac{\sin 2\theta}{\theta} &= \lim_{\theta \rightarrow 0} \left(\frac{\sin 2\theta}{2\theta} \cdot 2 \right) \\ &= 2 \lim_{t \rightarrow 0} \frac{\boxed{\sin t}}{t} \\ &= \boxed{2} \left(\because \lim_{t \rightarrow 0} \frac{\sin t}{t} = 1 \right) \end{aligned}$$

$$\begin{aligned} \text{[Sol 2]} \quad \lim_{\theta \rightarrow 0} \frac{\sin 2\theta}{\theta} &= \lim_{\theta \rightarrow 0} \frac{2 \sin \theta \cos \theta}{\theta} \\ &= \lim_{\theta \rightarrow 0} (2 \cos \theta) \cdot \lim_{\theta \rightarrow 0} \left(\frac{\boxed{\sin \theta}}{\theta} \right) \\ &= 2 \cdot 1 \cdot 1 \\ &= 2 \end{aligned}$$

3. Find the limit value of the following function, using either of the methods shown above.

$$\lim_{x \rightarrow 0} \frac{\sin 4x}{x}$$

[Sol 1] Letting $4x = t$, as $x \rightarrow 0$, $t \rightarrow 0$.

$$\begin{aligned} \therefore \lim_{x \rightarrow 0} \frac{\sin 4x}{x} &= \lim_{x \rightarrow 0} \left(\frac{\sin 4x}{4x} \cdot 4 \right) = 4 \lim_{t \rightarrow 0} \frac{\sin t}{t} \\ &= 4 \end{aligned}$$

$$\text{[Sol 2]} \quad \lim_{x \rightarrow 0} \frac{\sin 4x}{x} = \lim_{x \rightarrow 0} \frac{2 \sin 2x \cos 2x}{x}$$

$$= \lim_{x \rightarrow 0} (2 \cos 2x) \cdot \lim_{x \rightarrow 0} \frac{\sin 2x}{x} = 2 \cdot 1 \cdot 2 = 4$$

Solved
in previous problem

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	1	2	3	4

Find the limit values of the following functions.

Ex.

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{\tan x}{x} &= \lim_{x \rightarrow 0} \left(\frac{1}{x} \cdot \frac{\sin x}{\cos x} \right) \\
 &= \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \lim_{x \rightarrow 0} \frac{1}{\cos x} \\
 &= 1 \cdot 1 = 1
 \end{aligned}$$

$$\begin{aligned}
 (1) \lim_{x \rightarrow 0} \frac{\tan 4x}{x} &= \lim_{x \rightarrow 0} \left(\frac{1}{x} \cdot \frac{\sin 4x}{\cos 4x} \right) \\
 &= \lim_{x \rightarrow 0} \left(\frac{4}{4x} \cdot \frac{\sin 4x}{\cos 4x} \right) = \lim_{x \rightarrow 0} \frac{\sin 4x}{4x} \cdot \lim_{x \rightarrow 0} \frac{4}{\cos 4x} \\
 &= 1 \cdot 4 = 4
 \end{aligned}$$

$$\begin{aligned}
 (2) \lim_{\theta \rightarrow 0} \frac{\sin 3\theta}{\theta} &= \lim_{\theta \rightarrow 0} \left(\frac{\sin 3\theta}{3\theta} \cdot 3 \right) \\
 &= 3 \lim_{\theta \rightarrow 0} \frac{\sin 3\theta}{3\theta} = 3 \cdot 1 = 3
 \end{aligned}$$

$$\begin{aligned}
 (3) \lim_{x \rightarrow 0} \frac{1 - \cos 2x}{x^2} & \quad (\text{Hint}) \cos 2x = 1 - 2 \sin^2 x \\
 &= \lim_{x \rightarrow 0} \frac{1 - (1 - 2 \sin^2 x)}{x^2} \\
 &= \lim_{x \rightarrow 0} \frac{2 \sin^2 x}{x^2} = 2 \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^2 = 2 \cdot 1^2 = 2
 \end{aligned}$$

N 152 b

$$(4) \lim_{\theta \rightarrow 0} \frac{\sin 3\theta}{\sin 2\theta}$$

$$\begin{aligned} &= \lim_{\theta \rightarrow 0} \left(\frac{\sin 3\theta}{3\theta} \cdot \frac{2\theta}{\sin 2\theta} \cdot \frac{3}{2} \right) \\ &= \lim_{\theta \rightarrow 0} \left(\frac{\sin 3\theta}{3\theta} \cdot \frac{1}{\frac{\sin 2\theta}{2\theta}} \cdot \frac{3}{2} \right) \\ &= 1 \cdot \frac{1}{1} \cdot \frac{3}{2} \\ &= \frac{3}{2} \end{aligned}$$

$$(5) \lim_{x \rightarrow 0} \frac{\sin x^3}{x^2}$$

$$\begin{aligned} &= \lim_{x \rightarrow 0} \left(\frac{\sin x^3}{x^3} \cdot x \right) \\ &= \lim_{x \rightarrow 0} \frac{\sin x^3}{x^3} \cdot \lim_{x \rightarrow 0} x \\ &= 1 \cdot 0 = 0 \end{aligned}$$

$$(6) \lim_{x \rightarrow 0} \frac{\sin 3x}{\tan 2x}$$

$$\begin{aligned} &= \lim_{x \rightarrow 0} \frac{\sin 3x}{\sin 2x \cdot \frac{1}{\cos 2x}} = \lim_{x \rightarrow 0} \frac{\frac{\sin 3x}{3x} \cdot 3x}{\frac{\sin 2x}{2x} \cdot 2x \cdot \frac{1}{\cos 2x}} \\ &= \lim_{x \rightarrow 0} \frac{\frac{\sin 3x}{3x} \cdot 3}{\frac{\sin 2x}{2x} \cdot 2 \cdot \frac{1}{\cos 2x}} \\ &= \frac{3}{2} \end{aligned}$$

$$(7) \lim_{x \rightarrow \infty} \left(x \sin \frac{1}{x} \right)$$

[Sol] Let $\frac{1}{x} = t$. As $x \rightarrow \infty$, $t \rightarrow \boxed{0}$.

$$\therefore \lim_{x \rightarrow \infty} \left(x \sin \frac{1}{x} \right) = \lim_{t \rightarrow 0} \left(\frac{1}{t} \sin t \right) = \lim_{t \rightarrow 0} \frac{\sin t}{t} = 1$$

Time : to : Date Name

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(minutes) 0	-	1	-	2

Find the limit values of the following functions.

$$(1) \lim_{x \rightarrow 0} \frac{\sin x}{x + \sin x}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{\sin x}{x}}{1 + \frac{\sin x}{x}} = \boxed{\frac{1}{2}}$$

$$(2) \lim_{x \rightarrow 0} \frac{\sin 3x}{x + \sin x}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{\sin 3x}{x}}{1 + \frac{\sin x}{x}}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{\sin 3x}{3x} \cdot 3}{1 + \frac{\sin x}{x}} = \frac{3}{2}$$

$$(3) \lim_{\theta \rightarrow 0} \frac{\tan^2 \theta}{1 - \cos \theta}$$

$$= \lim_{\theta \rightarrow 0} \frac{\boxed{\sin^2 \theta}}{\cos^2 \theta (1 - \cos \theta)}$$

$$= \lim_{\theta \rightarrow 0} \frac{1 - \cos^2 \theta}{\cos^2 \theta (1 - \cos \theta)}$$

$$= \lim_{\theta \rightarrow 0} \frac{(1 + \cos \theta)(1 - \cos \theta)}{\cos^2 \theta (1 - \cos \theta)}$$

$$= \lim_{\theta \rightarrow 0} \frac{1 + \cos \theta}{\cos^2 \theta}$$

$$= \frac{1 + 1}{1^2}$$

$$= 2$$

N 153 b

$$(4) \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$$

[Sol 1]

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} &= \lim_{x \rightarrow 0} \frac{(1 - \cos x)(1 + \cos x)}{x^2(1 + \cos x)} \\ &= \lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2(1 + \cos x)} \\ &= \lim_{x \rightarrow 0} \left[\left(\frac{\sin x}{x} \right)^2 \cdot \frac{1}{1 + \cos x} \right] \\ &= 1^2 \cdot \frac{1}{1 + 1} \\ &= \frac{1}{2} \end{aligned}$$

[Sol 2]

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} &= \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{x}{2}}{x^2} \\ &= \lim_{x \rightarrow 0} \left[\left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 \cdot \frac{1}{2} \right] \\ &= \frac{1}{2} \end{aligned}$$

$$(5) \lim_{x \rightarrow 0} \frac{x \sin x}{1 - \cos x} \quad (\text{Use either of the methods shown above.})$$

[Sol 1]

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{x \sin x}{1 - \cos x} &= \lim_{x \rightarrow 0} \frac{x \sin x (1 + \cos x)}{(1 - \cos x)(1 + \cos x)} \\ &= \lim_{x \rightarrow 0} \frac{x \sin x (1 + \cos x)}{1 - \cos^2 x} \\ &= \lim_{x \rightarrow 0} \frac{x \sin x (1 + \cos x)}{\sin^2 x} \\ &= \lim_{x \rightarrow 0} \left[\frac{x}{\sin x} \cdot (1 + \cos x) \right] \\ &= 2 \end{aligned}$$

[Sol 2]

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{x \sin x}{1 - \cos x} &= \lim_{x \rightarrow 0} \frac{x \sin x}{2 \sin^2 \frac{x}{2}} \\ &= \lim_{x \rightarrow 0} \left[\left(\frac{\frac{x}{2}}{\sin \frac{x}{2}} \right)^2 \cdot \frac{\sin x}{x} \cdot 2 \right] \\ &= 2 \end{aligned}$$

Limits of Trigonometric Functions

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	1	-	2

Find the limit values of the following functions.

(1) $\lim_{x \rightarrow 0} \left(x \sin \frac{1}{x} \right)$

[Sol 1] Since $-1 \leq \sin \frac{1}{x} \leq 1$,(a) When $x > 0$

$$-x \leq x \sin \frac{1}{x} \leq x$$

(b) When $x < 0$,

$$x \leq x \sin \frac{1}{x} \leq -x$$

As $x \rightarrow 0$, $-x \rightarrow 0$.

$$\therefore \lim_{x \rightarrow 0} \left(x \sin \frac{1}{x} \right) = 0$$

[Sol 2] Since $\left| \sin \frac{1}{x} \right| \leq 1$,

$$\left| x \sin \frac{1}{x} \right| \leq |x|$$

Also, $\left| x \sin \frac{1}{x} \right| \geq 0$

Combining both conditions,

$$0 \leq \left| x \sin \frac{1}{x} \right| \leq |x|$$

As $x \rightarrow 0$, $|x| \rightarrow 0$.

$$\therefore \lim_{x \rightarrow 0} \left(x \sin \frac{1}{x} \right) = 0$$

(2) $\lim_{x \rightarrow 0} \left(x \cos \frac{1}{x} \right)$ (Use either of the methods shown above.)

[Sol 1] Since $-1 \leq \cos \frac{1}{x} \leq 1$,(a) When $x > 0$

$$-x \leq x \cos \frac{1}{x} \leq x$$

(b) When $x < 0$,

$$x \leq x \cos \frac{1}{x} \leq -x$$

As $x \rightarrow 0$, $-x \rightarrow 0$.

$$\therefore \lim_{x \rightarrow 0} \left(x \cos \frac{1}{x} \right) = 0$$

[Sol 2] Since $\left| \cos \frac{1}{x} \right| \leq 1$,

$$\left| x \cos \frac{1}{x} \right| \leq |x|$$

Also, $\left| x \cos \frac{1}{x} \right| \geq 0$

Combining both conditions,

$$0 \leq \left| x \cos \frac{1}{x} \right| \leq |x|$$

As $x \rightarrow 0$, $|x| \rightarrow 0$.

$$\therefore \lim_{x \rightarrow 0} \left(x \cos \frac{1}{x} \right) = 0$$

N 154 b

$$(3) \lim_{\theta \rightarrow \infty} \frac{\sin \theta}{\theta}$$

[Sol 1] Since $-1 \leq \sin \theta \leq 1$,

$$\begin{aligned} \text{As } \theta \rightarrow \infty, \\ -\frac{1}{\theta} \leq \frac{\sin \theta}{\theta} \leq \frac{1}{\theta}, \text{ and} \\ -\frac{1}{\theta}, \frac{1}{\theta} \rightarrow 0 \end{aligned}$$

$$\therefore \lim_{\theta \rightarrow \infty} \frac{\sin \theta}{\theta} = 0$$

$$(4) \lim_{\theta \rightarrow \infty} \frac{\cos \theta}{\theta}$$

[Sol 1] Since $-1 \leq \cos \theta \leq 1$,

$$\begin{aligned} \text{As } \theta \rightarrow \infty, \\ -\frac{1}{\theta} \leq \frac{\cos \theta}{\theta} \leq \frac{1}{\theta}, \text{ and} \\ -\frac{1}{\theta}, \frac{1}{\theta} \rightarrow 0 \end{aligned}$$

$$\therefore \lim_{\theta \rightarrow \infty} \frac{\cos \theta}{\theta} = 0$$

$$(5) \lim_{x \rightarrow \infty} \frac{x}{x + \sin x}$$

$$[\text{Sol 1}] \quad \frac{x}{x + \sin x} = \frac{1}{1 + \frac{\sin x}{x}}$$

Since $-1 \leq \sin x \leq 1$,

$$\begin{aligned} \text{As } x \rightarrow \infty, \\ -\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}, \text{ and} \\ -\frac{1}{x}, \frac{1}{x} \rightarrow 0 \end{aligned}$$

$$\therefore \lim_{x \rightarrow \infty} \frac{x}{x + \sin x} = \frac{1}{1 + 0} = 1$$

$$[\text{Sol 2}] \quad \text{Since } |\sin \theta| \leq 1, \left| \frac{\sin \theta}{\theta} \right| \leq \left| \frac{1}{\theta} \right|$$

$$\text{Also, } \left| \frac{\sin \theta}{\theta} \right| \geq 0$$

Combining both conditions,

$$0 \leq \left| \frac{\sin \theta}{\theta} \right| \leq \left| \frac{1}{\theta} \right|$$

$$\text{As } \theta \rightarrow \infty, \left| \frac{1}{\theta} \right| \rightarrow 0.$$

$$\therefore \lim_{\theta \rightarrow \infty} \frac{\sin \theta}{\theta} = 0$$

$$[\text{Sol 2}] \quad \text{Since } |\cos \theta| \leq 1, \left| \frac{\cos \theta}{\theta} \right| \leq \left| \frac{1}{\theta} \right|$$

$$\text{Also, } \left| \frac{\cos \theta}{\theta} \right| \geq 0$$

Combining both conditions,

$$0 \leq \left| \frac{\cos \theta}{\theta} \right| \leq \left| \frac{1}{\theta} \right|$$

$$\text{As } \theta \rightarrow \infty, \left| \frac{1}{\theta} \right| \rightarrow 0.$$

$$\therefore \lim_{\theta \rightarrow \infty} \frac{\cos \theta}{\theta} = 0$$

$$[\text{Sol 2}] \quad \frac{x}{x + \sin x} = \frac{1}{1 + \frac{\sin x}{x}}$$

$$\text{Since } |\sin x| \leq 1, \left| \frac{\sin x}{x} \right| \leq \left| \frac{1}{x} \right|$$

$$\text{Also, } \left| \frac{\sin x}{x} \right| \geq 0$$

Combining both conditions,

$$0 \leq \left| \frac{\sin x}{x} \right| \leq \left| \frac{1}{x} \right|$$

$$\text{As } x \rightarrow \infty, \left| \frac{1}{x} \right| \rightarrow 0.$$

$$\therefore \lim_{x \rightarrow \infty} \frac{x}{x + \sin x} = \frac{1}{1 + 0} = 1$$

Limits of Trigonometric Functions

Time : to : Date Name

100%	90%	80%	70%	60%~
(mistakes) 0	-	-	1	2~

Find the limit values of the following functions.

Ex. $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\pi - 2x}{\sin\left(x - \frac{\pi}{2}\right)}$

[Sol] Letting $\theta = x - \frac{\pi}{2}$, $\therefore x = \theta + \frac{\pi}{2}$

As $x \rightarrow \frac{\pi}{2}$, $\theta \rightarrow 0$.

$$\therefore \lim_{x \rightarrow \frac{\pi}{2}} \frac{\pi - 2x}{\sin\left(x - \frac{\pi}{2}\right)} = \lim_{\theta \rightarrow 0} \frac{-2\theta}{\sin \theta}$$
$$= -2$$

(1) $\lim_{x \rightarrow \frac{\pi}{3}} \frac{\pi - 3x}{\tan\left(x - \frac{\pi}{3}\right)}$

[Sol] Letting $\theta = x - \frac{\pi}{3}$, $\therefore x = \theta + \frac{\pi}{3}$

As $x \rightarrow \frac{\pi}{3}$, $\theta \rightarrow 0$.

$$\therefore \lim_{x \rightarrow \frac{\pi}{3}} \frac{\pi - 3x}{\tan\left(x - \frac{\pi}{3}\right)} = \lim_{\theta \rightarrow 0} \left(\frac{-3\theta}{\sin \theta} \cdot \cos \theta \right)$$
$$= -3$$

(2) $\lim_{2x \rightarrow \frac{\pi}{2}} \frac{3\pi - 12x}{\sin(4x - \pi)}$

[Sol] Letting $\theta = 2x - \frac{\pi}{2}$, $\therefore x = \frac{\theta}{2} + \frac{\pi}{4}$

As $2x \rightarrow \frac{\pi}{2}$, $\theta \rightarrow 0$.

$$\therefore \lim_{2x \rightarrow \frac{\pi}{2}} \frac{3\pi - 12x}{\sin(4x - \pi)} = \lim_{\theta \rightarrow 0} \frac{-6\theta}{\sin 2\theta}$$
$$= -3$$

N 155 b

$$(3) \lim_{x \rightarrow \frac{\pi}{2}} [(\pi - 2x) \tan x]$$

[Sol] Letting $\theta = \frac{\pi}{2} - x$, $x = \frac{\pi}{2} - \theta$

As $x \rightarrow \frac{\pi}{2}$, $\theta \rightarrow 0$.

$$\begin{aligned} \therefore \lim_{x \rightarrow \frac{\pi}{2}} [(\pi - 2x) \tan x] &= \lim_{\theta \rightarrow 0} \left[2\theta \cdot \tan \left(\frac{\pi}{2} - \theta \right) \right] \\ &= \lim_{\theta \rightarrow 0} \left(2\theta \cdot \frac{1}{\tan \theta} \right) \\ &= \lim_{\theta \rightarrow 0} \left(2\theta \cdot \frac{\cos \theta}{\sin \theta} \right) \\ &= 2 \lim_{\theta \rightarrow 0} \left(\frac{\theta}{\sin \theta} \cdot \cos \theta \right) \\ &= 2 \cdot 1 \cdot 1 \\ &= 2 \end{aligned}$$

$$(4) \lim_{x \rightarrow 1} \frac{1 + \cos(\pi x)}{(x - 1)^2}$$

[Sol] Letting $\theta = x - 1$, $x = \theta + 1$

As $x \rightarrow 1$, $\theta \rightarrow 0$.

$$\begin{aligned} \therefore \lim_{x \rightarrow 1} \frac{1 + \cos(\pi x)}{(x - 1)^2} &= \lim_{\theta \rightarrow 0} \frac{1 + \cos[\pi(\theta + 1)]}{\theta^2} = \lim_{\theta \rightarrow 0} \frac{1 + \cos(\pi\theta + \pi)}{\theta^2} \\ &= \lim_{\theta \rightarrow 0} \frac{1 - \cos(\pi\theta)}{\theta^2} \\ &= \lim_{\theta \rightarrow 0} \frac{(1 - \cos \pi\theta)(1 + \cos \pi\theta)}{\theta^2(1 + \cos \pi\theta)} \\ &= \lim_{\theta \rightarrow 0} \frac{1 - \cos^2 \pi\theta}{\theta^2(1 + \cos \pi\theta)} \\ &= \lim_{\theta \rightarrow 0} \left(\frac{\sin^2 \pi\theta}{\pi^2 \theta^2} \cdot \pi^2 \cdot \frac{1}{1 + \cos \pi\theta} \right) \\ &= \frac{\pi^2}{2} \end{aligned}$$

Limits of Trigonometric Functions

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	—	—	—	—

Find the limit value of the following functions.

$$(1) \lim_{x \rightarrow 0} \frac{\sin(\sin x)}{x}$$

[Sol] Letting $\theta = \sin x$,

$$\text{As } x \rightarrow 0, \quad \theta \rightarrow 0.$$

$$\therefore \lim_{x \rightarrow 0} \frac{\sin(\sin x)}{x} = \lim_{x \rightarrow 0} \left[\frac{\sin(\sin x)}{\sin x} \cdot \frac{\sin x}{x} \right]$$

$$= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \cdot \lim_{x \rightarrow 0} \frac{\sin x}{x}$$

$$= 1 \cdot 1$$

$$= 1$$

$$(2) \lim_{x \rightarrow 0} \frac{\tan(\sin \pi x)}{x}$$

[Sol] Letting $\theta = \boxed{\pi x}$, $x = \frac{\theta}{\pi}$

$$\text{As } x \rightarrow 0, \quad \theta \rightarrow 0.$$

$$\therefore \lim_{x \rightarrow 0} \frac{\tan(\sin \pi x)}{x} = \lim_{\theta \rightarrow 0} \frac{\tan(\sin \theta)}{\frac{\theta}{\pi}}$$

$$= \lim_{\theta \rightarrow 0} \left[\frac{\sin(\sin \theta)}{\theta} \cdot \frac{\pi}{\cos(\sin \theta)} \right]$$

$$= \lim_{\theta \rightarrow 0} \left[\frac{\sin(\sin \theta)}{\sin \theta} \cdot \frac{\sin \theta}{\theta} \cdot \frac{\pi}{\cos(\sin \theta)} \right]$$

$$= 1 \cdot 1 \cdot \frac{\pi}{\cos 0}$$

$$= \pi$$

N 156 b

$$(3) \lim_{x \rightarrow 0} \frac{1 - \cos(1 - \cos x)}{x^4}$$

$$= \lim_{x \rightarrow 0} \frac{[1 - \cos(1 - \cos x)] \cdot [1 + \cos(1 - \cos x)]}{x^4 [1 + \cos(1 - \cos x)]}$$

$$= \lim_{x \rightarrow 0} \frac{1 - \cos^2(1 - \cos x)}{x^4 [1 + \cos(1 - \cos x)]}$$

$$= \lim_{x \rightarrow 0} \frac{\sin^2(1 - \cos x)}{x^4 [1 + \cos(1 - \cos x)]}$$

$$= \lim_{x \rightarrow 0} \left\{ \left[\frac{\sin(1 - \cos x)}{(1 - \cos x)} \right]^2 \cdot (1 - \cos x)^2 \cdot \frac{1}{x^4} \cdot \frac{1}{1 + \cos(1 - \cos x)} \right\}$$

$$= \lim_{x \rightarrow 0} \left\{ \left[\frac{\sin(1 - \cos x)}{(1 - \cos x)} \right]^2 \cdot \frac{1}{1 + \cos(1 - \cos x)} \right\} \cdot \lim_{x \rightarrow 0} \frac{(1 - \cos x)^2}{x^4}$$

$$= 1 \cdot \frac{1}{2} \cdot \lim_{x \rightarrow 0} \frac{(1 - \cos x)^2 (1 + \cos x)^2}{x^4 (1 + \cos x)^2}$$

$$= \frac{1}{2} \cdot \lim_{x \rightarrow 0} \frac{(1 - \cos^2 x)^2}{x^4 (1 + \cos x)^2}$$

$$= \frac{1}{2} \cdot \lim_{x \rightarrow 0} \frac{\sin^4 x}{x^4 (1 + \cos x)^2}$$

$$= \frac{1}{2} \cdot \lim_{x \rightarrow 0} \left[\frac{\sin^4 x}{x^4} \cdot \frac{1}{(1 + \cos x)^2} \right]$$

$$= \frac{1}{2} \cdot 1 \cdot \frac{1}{(1 + 1)^2}$$

$$= \frac{1}{8}$$

$$\therefore \lim_{x \rightarrow 0} \frac{1 - \cos(1 - \cos x)}{x^4} = \frac{1}{8}$$

Limits of Trigonometric Functions

Time : to : Date Name

100%	90%	80%	70%	60%~
(mistakes) 0	-	1	-	2-

Find the limit values of the following functions.

$$\begin{aligned}
 (1) \quad & \lim_{x \rightarrow 0} \frac{\sqrt{1+x \sin x} - 1}{x^2} \\
 &= \lim_{x \rightarrow 0} \frac{(\sqrt{1+x \sin x} - 1)(\sqrt{1+x \sin x} + 1)}{x^2(\sqrt{1+x \sin x} + 1)} \\
 &= \lim_{x \rightarrow 0} \frac{(1+x \sin x) - 1}{x^2(\sqrt{1+x \sin x} + 1)} \\
 &= \lim_{x \rightarrow 0} \frac{\sin x}{x(\sqrt{1+x \sin x} + 1)} \\
 &= \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \cdot \frac{1}{\sqrt{1+x \sin x} + 1} \right) \\
 &= 1 \cdot \frac{1}{1+1} = \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & \lim_{x \rightarrow 0} \frac{\sqrt{2-\cos x} - 1}{3x^2} \\
 &= \lim_{x \rightarrow 0} \frac{(\sqrt{2-\cos x} - 1)(\sqrt{2-\cos x} + 1)}{3x^2(\sqrt{2-\cos x} + 1)} \\
 &= \lim_{x \rightarrow 0} \frac{(2-\cos x) - 1}{3x^2(\sqrt{2-\cos x} + 1)} \\
 &= \lim_{x \rightarrow 0} \frac{1-\cos x}{3x^2(\sqrt{2-\cos x} + 1)} \\
 &= \lim_{x \rightarrow 0} \frac{(1-\cos x)(1+\cos x)}{3x^2(\sqrt{2-\cos x} + 1)(1+\cos x)} \\
 &= \lim_{x \rightarrow 0} \frac{1-\cos^2 x}{3x^2(\sqrt{2-\cos x} + 1)(1+\cos x)} \\
 &= \lim_{x \rightarrow 0} \frac{\sin^2 x}{3x^2(\sqrt{2-\cos x} + 1)(1+\cos x)} \\
 &= \frac{1}{3 \cdot 2 \cdot 2} = \frac{1}{12}
 \end{aligned}$$

N 157 b

$$\begin{aligned}
 (3) \quad \lim_{x \rightarrow 0} \frac{\tan x(1 - \cos x)}{x^3} &= \lim_{x \rightarrow 0} \frac{\sin x(1 - \cos x)}{(\cos x)x^3} \\
 &= \lim_{x \rightarrow 0} \frac{\sin x(1 - \cos x)(1 + \cos x)}{x^3(\cos x)(1 + \cos x)} \\
 &= \lim_{x \rightarrow 0} \frac{\sin x(1 - \cos^2 x)}{x^3(\cos x)(1 + \cos x)} \\
 &= \lim_{x \rightarrow 0} \frac{\sin^3 x}{x^3(\cos x)(1 + \cos x)} \\
 &= \frac{1}{1+1} = \frac{1}{2}
 \end{aligned}$$

$$(4) \quad \lim_{t \rightarrow 2\pi} \frac{\sin t}{t^2 - 4\pi^2}$$

[Sol] Letting $\theta = t - 2\pi$, $t = \theta + 2\pi$

As $t \rightarrow 2\pi$, $\theta \rightarrow 0$.

$$\begin{aligned}
 \lim_{t \rightarrow 2\pi} \frac{\sin t}{t^2 - 4\pi^2} &= \lim_{\theta \rightarrow 0} \frac{\sin(\theta + 2\pi)}{(\theta + 2\pi)^2 - 4\pi^2} = \lim_{\theta \rightarrow 0} \frac{\sin(\theta + 2\pi)}{\theta(\theta + 4\pi)} \\
 &= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta(\theta + 4\pi)} \quad \left(\sin(\theta + 2\pi) = \sin \theta \right) \\
 &= \lim_{\theta \rightarrow 0} \left(\frac{\sin \theta}{\theta} \cdot \frac{1}{\theta + 4\pi} \right) \\
 &= 1 \cdot \frac{1}{4\pi} = \frac{1}{4\pi}
 \end{aligned}$$

$$(5) \quad \lim_{x \rightarrow 0} \frac{1 - \cos ax}{x^2} \quad (a \neq 0)$$

$$\begin{aligned}
 &= \lim_{x \rightarrow 0} \frac{(1 - \cos ax)(1 + \cos ax)}{x^2(1 + \cos ax)} \\
 &= \lim_{x \rightarrow 0} \frac{\sin^2 ax}{x^2(1 + \cos ax)} \\
 &= \lim_{x \rightarrow 0} \left(\frac{\sin^2 ax}{a^2 x^2} \cdot a^2 \cdot \frac{1}{1 + \cos ax} \right) \\
 &= 1 \cdot a^2 \cdot \frac{1}{1+1} = \frac{a^2}{2}
 \end{aligned}$$

Limits of Trigonometric Functions

Time : to : Date Name

100%	90%	80%	70%	60%~
(mistake) 0	-	-	-	1-

1. Find the limit value of the following function as $x \rightarrow 0$.

$$f(x) = \frac{\tan x - \sin x}{x^n} \quad (\text{where } n \text{ is a natural number})$$

(Analyze the cases when $n \leq 2$, $n = 3$ and $n \geq 4$.)

$$\begin{aligned}
 [\text{Sol}] f(x) &= \frac{\tan x - \sin x}{x^n} = \frac{1}{x^n} \cdot \sin x \left(\frac{1}{\cos x} - 1 \right) \\
 &= \frac{\sin x (1 - \cos x)}{x^n \cos x} \\
 &= \frac{\sin x (1 - \cos x)(1 + \cos x)}{x^n \cos x (1 + \cos x)} \\
 &= \frac{\sin x (1 - \cos^2 x)}{x^n \cos x (1 + \cos x)} = \frac{\sin^3 x}{x^n \cos x (1 + \cos x)} \\
 &= \frac{1}{x^{n-3}} \cdot \left(\frac{\sin x}{x} \right)^3 \cdot \frac{1}{\cos x (1 + \cos x)}
 \end{aligned}$$

(a) When $n \leq 2$,

$$\begin{aligned}
 \lim_{x \rightarrow 0} f(x) &= \lim_{x \rightarrow 0} \left[x^{3-n} \cdot \left(\frac{\sin x}{x} \right)^3 \cdot \frac{1}{\cos x (1 + \cos x)} \right] \\
 &= 0
 \end{aligned}$$

(b) When $n = 3$,

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \left[\left(\frac{\sin x}{x} \right)^3 \cdot \frac{1}{\cos x (1 + \cos x)} \right] = \frac{1}{2}$$

(c) When $n \geq 4$,

$$\text{When } n \text{ is odd, } \lim_{x \rightarrow 0} f(x) = \infty,$$

$$\text{When } n \text{ is even, no limit exists. (because } \lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^-} f(x))$$

N 158 b

2. Find the limit value of the following function as $x \rightarrow 0$.

$$f(x) = \frac{\sin^2 x - \tan^2 x}{x^n} \quad (\text{where } n \text{ is a natural number})$$

$$\begin{aligned} [\text{Sol}] f(x) &= \frac{\sin^2 x - \tan^2 x}{x^n} = \frac{1}{x^n} \cdot \sin^2 x \left(1 - \frac{1}{\cos^2 x} \right) \\ &= \frac{\sin^2 x (\cos^2 x - 1)}{x^n \cos^2 x} \\ &= -\frac{\sin^4 x}{x^n \cos^2 x} \\ &= -\frac{1}{x^{n-4}} \cdot \left(\frac{\sin x}{x} \right)^4 \cdot \frac{1}{\cos^2 x} \end{aligned}$$

(a) When $n \leq 3$,

$$\begin{aligned} \lim_{x \rightarrow 0} f(x) &= -\lim_{x \rightarrow 0} \left[x^{4-n} \cdot \left(\frac{\sin x}{x} \right)^4 \cdot \frac{1}{\cos^2 x} \right] \\ &= 0 \end{aligned}$$

b) When $n = 4$,

$$\begin{aligned} \lim_{x \rightarrow 0} f(x) &= -\lim_{x \rightarrow 0} \left[\left(\frac{\sin x}{x} \right)^4 \cdot \frac{1}{\cos^2 x} \right] \\ &= -1 \end{aligned}$$

c) When $n \geq 5$,

When n is odd, no limit exists. (because $\lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^-} f(x)$)

When n is even, $\lim_{x \rightarrow 0} f(x) = -\infty$.

Limits of Trigonometric Functions

Time : to : Date Name

100% (mistakes) 0	90%	80%	70%	60%~
-	-	-	-	1-

1. Find the conditions for the constants a , b , p and q at which the function $f(x) = \frac{(px + q)\sin 2x}{ax + b}$ satisfies the following equalities.

$$\lim_{x \rightarrow 0} f(x) = 2 \quad \dots \textcircled{1} \quad \text{and} \quad \lim_{x \rightarrow \infty} f(x) = 0 \quad \dots \textcircled{2}$$

[Sol] Since $\lim_{x \rightarrow 0} [(px + q)\sin 2x] = 0$,

the limit of the denominator must also be equal to zero.

$$\therefore \lim_{x \rightarrow 0} (ax + b) = 0$$

Therefore, $b = \boxed{0}$.

Also, since the denominator is $ax + b$, $a \neq 0$

Substituting this back into the original function,

$$\begin{aligned} \lim_{x \rightarrow 0} f(x) &= \lim_{x \rightarrow 0} \frac{(px + q)\sin 2x}{ax} \\ &= \lim_{x \rightarrow 0} \left(\frac{px + q}{\boxed{a}} \cdot \frac{\sin 2x}{2x} \cdot 2 \right) = \boxed{\frac{2q}{a}} \quad \dots \textcircled{3} \end{aligned}$$

From $\textcircled{1}$ and $\textcircled{3}$, $q = \boxed{a} \neq 0$

$$\text{Therefore, } f(x) = \frac{(px + a)\sin 2x}{ax} = \frac{p}{a} \sin 2x + \frac{\sin 2x}{x}$$

$$\text{Since } \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \left(\frac{p}{a} \sin 2x + \frac{\sin 2x}{x} \right),$$

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \left(\frac{p}{a} \sin 2x \right) + \lim_{x \rightarrow \infty} \frac{\sin 2x}{x}$$

$$\text{Since } 0 \leq |\sin 2x| \leq 1, \quad 0 \leq \left| \frac{\sin 2x}{x} \right| \leq \left| \frac{1}{x} \right|$$

$$\text{As } x \rightarrow \infty, \quad \left| \frac{1}{x} \right| \rightarrow 0, \quad \therefore \lim_{x \rightarrow \infty} \frac{\sin 2x}{x} = 0$$

As $x \rightarrow \infty$, in order for $f(x) \rightarrow 0$, $p = \boxed{0}$.

Answer: $\boxed{a} = \boxed{q} \neq 0$, $\boxed{b} = \boxed{p} = 0$

N 159 b

2. Find the conditions for the constants a , b , p , and q at which the function $f(x) = \frac{(px^2 - q)\sin^2 3x}{ax^2 - b}$ satisfies the following equalities.

$$\lim_{x \rightarrow 0} f(x) = 3 \quad \dots \textcircled{1} \quad \text{and} \quad \lim_{x \rightarrow \infty} f(x) = 0 \quad \dots \textcircled{2}$$

[Sol] Since $\lim_{x \rightarrow 0} [(px^2 - q)\sin^2 3x] = 0$,

the limit of the denominator must also be equal to zero.

$$\therefore \lim_{x \rightarrow 0} (ax^2 - b) = 0 \quad \text{Therefore, } b = 0$$

Also, since the denominator is $ax^2 - b$, $a \neq 0$

Substituting this back into the original function,

$$\begin{aligned} \lim_{x \rightarrow 0} f(x) &= \lim_{x \rightarrow 0} \frac{(px^2 - q)\sin^2 3x}{ax^2} \\ &= \lim_{x \rightarrow 0} \left[\frac{px^2 - q}{a} \cdot \left(\frac{\sin 3x}{3x} \right)^2 \cdot 9 \right] = -\frac{9q}{a} \quad \dots \textcircled{3} \end{aligned}$$

$$\text{From } \textcircled{1} \text{ and } \textcircled{3}, \quad q = -\frac{a}{3} \neq 0$$

$$\text{Therefore, } f(x) = \frac{\left(px^2 + \frac{a}{3} \right) \sin^2 3x}{ax^2} = \frac{p}{a} \cdot \sin^2 3x + 3 \left(\frac{\sin 3x}{3x} \right)^2$$

$$\text{Thus, } \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \left(\frac{p}{a} \cdot \sin^2 3x \right) + 3 \lim_{x \rightarrow \infty} \left(\frac{\sin 3x}{3x} \right)^2$$

$$\text{Since } 0 \leq |\sin 3x| \leq 1, \quad 0 \leq |\sin^2 3x| \leq 1, \quad 0 \leq \left| \frac{\sin^2 3x}{9x^2} \right| \leq \left| \frac{1}{9x^2} \right|$$

$$\text{As } x \rightarrow \infty, \quad \left| \frac{1}{9x^2} \right| \rightarrow 0. \quad \therefore \lim_{x \rightarrow \infty} \left(\frac{\sin 3x}{3x} \right)^2 = 0$$

As $x \rightarrow \infty$, in order for $f(x) \rightarrow 0$, $p = 0$.

$$\text{Answer: } -\frac{a}{3} = q \neq 0, \quad b = p = 0$$

Limits of Trigonometric Functions

Time : to : Date Name

100%	90%	80%	70%	60%~
(mistakes) 0	-	1	-	2-

Find the limit values of the following functions.

$$(1) \lim_{x \rightarrow 0} \frac{3 \sin 4x}{x + \sin x}$$

$$= 3 \lim_{x \rightarrow 0} \frac{\frac{\sin 4x}{x}}{1 + \frac{\sin x}{x}}$$

$$= 3 \lim_{x \rightarrow 0} \frac{\frac{\sin 4x}{4x} \cdot 4}{1 + \frac{\sin x}{x}} = 3 \cdot \frac{1 \cdot 4}{1 + 1} = 6$$

$$(2) \lim_{x \rightarrow 0} \frac{\tan 2x}{x}$$

$$= \lim_{x \rightarrow 0} \left(\frac{1}{x} \cdot \frac{\sin 2x}{\cos 2x} \right)$$

$$= \lim_{x \rightarrow 0} \left(\frac{2}{2x} \cdot \frac{\sin 2x}{\cos 2x} \right) = \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \cdot \lim_{x \rightarrow 0} \frac{2}{\cos 2x}$$
$$= 1 \cdot 2 = 2$$

$$(3) \lim_{x \rightarrow \infty} \left(x \tan \frac{3}{x} \right)$$

[Sol] Letting $\frac{3}{x} = t$, as $x \rightarrow \infty$, $t \rightarrow 0$

$$\lim_{x \rightarrow \infty} \left(x \tan \frac{3}{x} \right) = \lim_{t \rightarrow 0} \left(\frac{3}{t} \tan t \right)$$

$$= \lim_{t \rightarrow 0} \left(\frac{3}{t} \cdot \frac{\sin t}{\cos t} \right) = \lim_{t \rightarrow 0} \left(\frac{\sin t}{t} \cdot \frac{3}{\cos t} \right)$$
$$= 1 \cdot 3 = 3$$

N 160 b

$$(4) \lim_{2x \rightarrow \frac{\pi}{3}} \frac{3\pi - 18x}{\sin(6x - \pi)}$$

$$[\text{Sol}] \text{ Letting } \theta = 2x - \frac{\pi}{3}, \quad x = \frac{\theta}{2} + \frac{\pi}{6}$$

$$\text{As } 2x \rightarrow \frac{\pi}{3}, \quad \theta \rightarrow 0.$$

$$\begin{aligned} \therefore \lim_{2x \rightarrow \frac{\pi}{3}} \frac{3\pi - 18x}{\sin(6x - \pi)} &= \lim_{\theta \rightarrow 0} \frac{-9\theta}{\sin 3\theta} \\ &= -3 \end{aligned}$$

$$(5) \lim_{x \rightarrow 0} \frac{\tan\left(\sin \frac{\pi}{2} x\right)}{\frac{x}{2}}$$

$$[\text{Sol}] \text{ Letting } \theta = \frac{\pi}{2}x, \quad x = \frac{2\theta}{\pi}$$

$$\text{As } x \rightarrow 0, \quad \theta \rightarrow 0.$$

$$\therefore \lim_{x \rightarrow 0} \frac{\tan\left(\sin \frac{\pi}{2} x\right)}{\frac{x}{2}} = \lim_{\theta \rightarrow 0} \frac{\tan(\sin \theta)}{\frac{\theta}{\pi}}$$

$$= \lim_{\theta \rightarrow 0} \left[\frac{\sin(\sin \theta)}{\theta} \cdot \frac{\pi}{\cos(\sin \theta)} \right]$$

$$= \lim_{\theta \rightarrow 0} \left[\frac{\sin(\sin \theta)}{\sin \theta} \cdot \frac{\sin \theta}{\theta} \cdot \frac{\pi}{\cos(\sin \theta)} \right]$$

$$= 1 \cdot 1 \cdot \frac{\pi}{\cos 0}$$

$$= \pi$$

Continuous and Discontinuous Functions

Time : to : Date Name

100%	90%	80%	70%	69%
(mistakes) 0	-	-	-	1-

Most of the functions which we have dealt with up to now, such as $y = ax^2 + bx + c$ or $y = \sin x$, when drawn on a graph, form a curved line without breaking anywhere. They are called *continuous functions*.

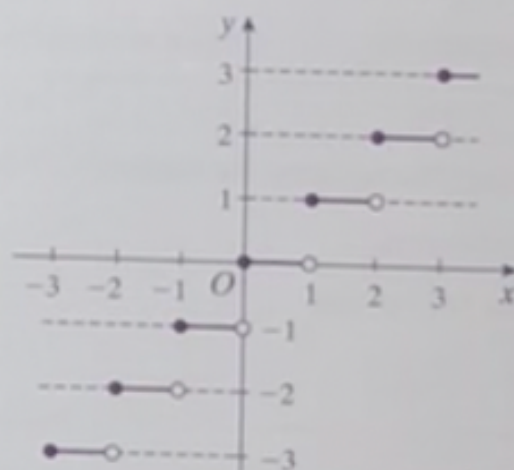
Let us look at the following function, which is called the *step function*.

$$f(x) = [x]$$

(Where $[x]$ represents the greatest integer less than or equal to x .)

x	-1.8	-1	-0.6	-0.3	0	0.5	0.7	1.1	1.8	2
$f(x) = [x]$	-2	-1	-1	-1	0	0	0	1	1	2

1. Using the graph of $f(x) = [x]$ shown below, investigate the point at $x = 1$.



From the graph, $f(1) = [1]$, $\lim_{x \rightarrow 1^+} f(x) = [1]$, and $\lim_{x \rightarrow 1^-} f(x) = [0]$.

Thus, $\lim_{x \rightarrow 1} f(x)$ does not exist.

Therefore, it is said that the function $f(x)$ is *discontinuous* at $x = 1$.

Similarly, $f(n) = [n]$, $\lim_{x \rightarrow n^+} f(x) = [n]$ and $\lim_{x \rightarrow n^-} f(x) = [n-1]$.

Thus, $\lim_{x \rightarrow n} f(x)$ does not exist.

From the above, we can conclude that:

The function $f(x)$ is discontinuous at $x = n$ (where n is an integer).

N 161 b

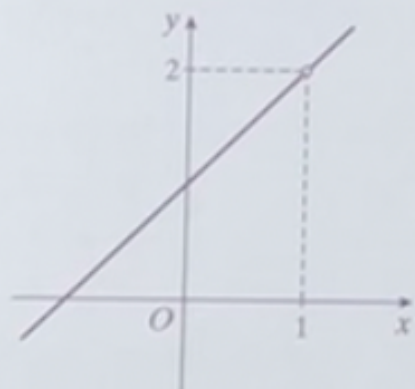
2. Examine the continuity of the function $f(x) = \frac{x^2 - 1}{x - 1}$.

[Sol] When $x \neq 1$, $f(x) = x + 1$.

Thus, $\lim_{x \rightarrow 1} (x + 1) = \boxed{2}$, and the value of the limit exists.

However, $f(1)$ does not exist.

Therefore, the function is discontinuous at $x = \boxed{1}$.



Conditions of function continuity:

When a function is continuous at $x = a$, the following three conditions must be fulfilled.

- (a) The value of $f(a)$ exists.
- (b) The value of $\lim_{x \rightarrow a} f(x)$ exists.
- (c) $\lim_{x \rightarrow a} f(x) = f(a)$

3. Using the above information, fill in the following blanks.

$f(x) = [x]$ is discontinuous at certain points because it does not fulfill conditions (b) and (c).

$f(x) = \frac{x^2 - 1}{x - 1}$ is discontinuous at $x = 1$ because it does not fulfill conditions $\boxed{(a)}$ and $\boxed{(c)}$.

Note:

- * When a function $f(x)$ is continuous over an interval, its graph forms a curve that does not break over that interval.
- * When a function $y = f(x)$ breaks at point $x = a$, $y = f(x)$ is said to be discontinuous at $x = a$.

Continuous and Discontinuous Functions

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	-	1	2

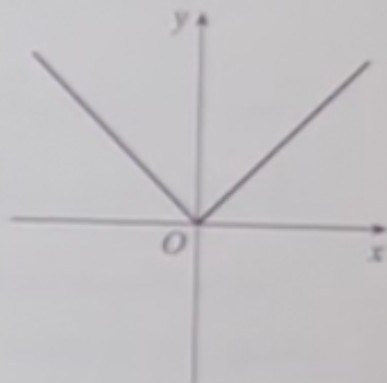
Examine the continuity of the following functions.

(Note: If a certain limit does not exist, write "does not exist.")

(1) $f(x) = |x|$

[Sol]

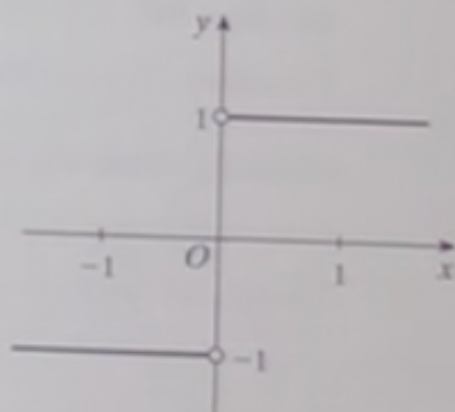
$$f(x) = \begin{cases} x & (x > 0) \\ 0 & (x = 0) \\ -x & (x < 0) \end{cases}$$

Thus, $f(x)$ is continuous when $x > 0$ and $x < 0$.Also, since $\lim_{x \rightarrow 0} |x| = 0 = f(0)$, this function is continuous at $x = 0$.Therefore, $f(x) = |x|$ is continuous over $-\infty < x < \infty$.

(2) $f(x) = \frac{|x|}{x}$

[Sol]

$$f(x) = \begin{cases} 1 & (x > 0) \\ -1 & (x < 0) \end{cases}$$

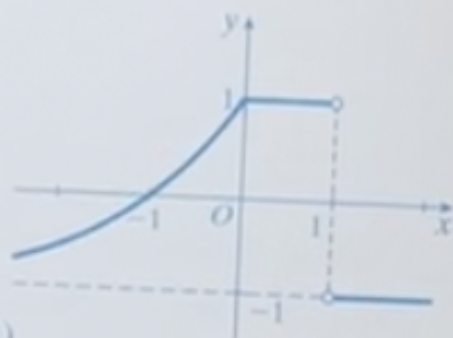
Thus, $f(x)$ is continuous when $x > 0$ and $x < 0$.Since $x = 0$ is not within the domain, $f(x)$ is discontinuous at $x = 0$ and continuous everywhere else.

N 162 b

$$(3) f(x) = \frac{1 - |x|}{|1 - x|}$$

[Sol]

$$f(x) = \begin{cases} \frac{1+x}{1-x} = -1 - \frac{2}{x-1} & (x < 0) \\ 1 & (0 < x < 1) \\ -1 & (1 < x) \end{cases}$$



Thus, $f(x)$ is continuous over $x < 0$, $0 < x < 1$ and $1 < x$.

As for $x = 0$, since $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{1 - |x|}{|1 - x|} = 1 = f(0)$,

$f(x)$ is also continuous at $x = 0$.

Since $f(1)$ does not exist, it is discontinuous at $x = 1$.

Therefore, $f(x)$ is discontinuous at $x = 1$ and continuous everywhere else.

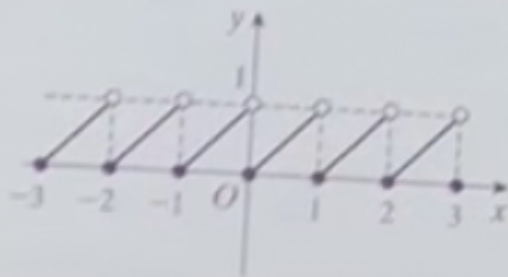
$$(4) f(x) = x - [x]$$

[Sol] Since $f(1) = 0$, and $\lim_{x \rightarrow 1^-} f(x) = 0$, $\lim_{x \rightarrow 1^+} f(x) = 1$,
we know that $\lim_{x \rightarrow 1} f(x)$ does not exist.

Therefore, $f(x)$ is discontinuous at $x = 1$.

Similarly, when n is an integer,

$f(n) = 0$, $\lim_{x \rightarrow n^-} f(x) = 0$, and $\lim_{x \rightarrow n^+} f(x) = 1$.
Therefore, $\lim_{x \rightarrow n} f(x)$ does not exist.



Thus, the function $f(x)$ is discontinuous at $x = n$ (where n is an integer) and continuous everywhere else.

Continuous and Discontinuous Functions

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	-	1	2-

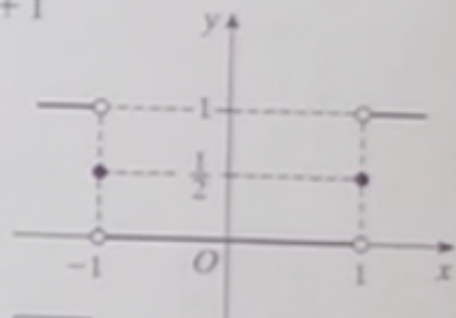
Draw the graphs of the functions defined by the following limits, and determine their continuity.

$$(1) f(x) = \lim_{n \rightarrow \infty} \frac{x^{2n}}{1 + x^{2n}}$$

[Sol] (a) When $|x| < 1$, $f(x) = \lim_{n \rightarrow \infty} \frac{x^{2n}}{1 + x^{2n}} = \boxed{0}$

(b) When $|x| > 1$, $f(x) = \lim_{n \rightarrow \infty} \frac{1}{\frac{1}{x^{2n}} + 1} = \boxed{1}$

(c) When $x = \pm 1$, $f(x) = \boxed{\frac{1}{2}}$



Therefore, $f(x)$ is discontinuous at $x = \boxed{\pm 1}$ and continuous everywhere else.

$$(2) f(x) = \lim_{n \rightarrow \infty} \frac{x^n - 1}{x^n + 1}$$

(Hint) Check $\lim_{x \rightarrow 1^-}$ and $\lim_{x \rightarrow 1^+}$.

[Sol] (a) When $|x| < 1$, $f(x) = \lim_{n \rightarrow \infty} \frac{x^n - 1}{x^n + 1} = -1$

(b) When $|x| > 1$, $f(x) = \lim_{n \rightarrow \infty} \frac{1 - \frac{1}{x^n}}{1 + \frac{1}{x^n}} = 1$

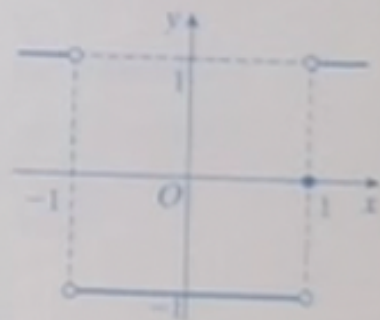
(c) When $x = 1$, $f(x) = 0$

However, $\lim_{x \rightarrow 1^-} f(x) = -1$,

and $\lim_{x \rightarrow 1^+} f(x) = 1$

Thus, $\lim_{x \rightarrow 1} f(x)$ does not exist.

(d) When $x = -1$, $f(x)$ does not exist.



Therefore, the graph is discontinuous at $x = \pm 1$ and continuous everywhere else.

$$(3) f(x) = \lim_{n \rightarrow \infty} \frac{|x|^n - 1}{|x|^n + 1} + x$$

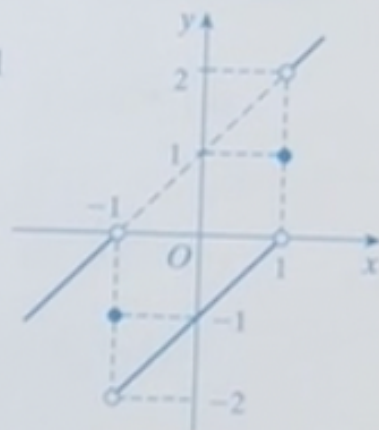
[Sol]

$$(a) \text{ When } |x| < 1, \quad f(x) = x - 1$$

$$(b) \text{ When } |x| > 1, \quad f(x) = \lim_{n \rightarrow \infty} \frac{1 - \frac{1}{|x|^n}}{1 + \frac{1}{|x|^n}} + x = x + 1$$

$$(c) \text{ When } x = 1, \quad f(x) = f(1) = 1$$

$$(d) \text{ When } x = -1, \quad f(x) = f(-1) = -1$$



Therefore, the graph of $y = f(x)$ is discontinuous at $x = \pm 1$ and continuous everywhere else.

$$(4) f(x) = x \lim_{n \rightarrow \infty} \frac{1 - x^n}{1 + x^n}$$

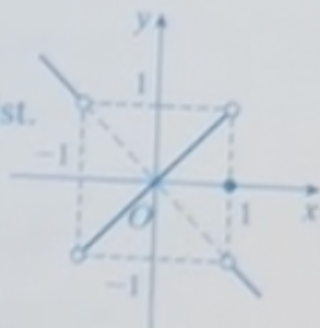
[Sol]

$$(a) \text{ When } |x| < 1, \quad f(x) = x$$

$$(b) \text{ When } |x| > 1, \quad f(x) = x \lim_{n \rightarrow \infty} \frac{\frac{1}{x^n} - 1}{\frac{1}{x^n} + 1} = -x$$

$$(c) \text{ When } x = 1, \quad f(x) = f(1) = 0$$

$$(d) \text{ When } x = -1, \quad f(x) = f(-1) \text{ does not exist.}$$



Therefore, the graph of $y = f(x)$ is discontinuous at $x = \pm 1$ and continuous everywhere else.

Continuous and Discontinuous Functions

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	-	-	-

Draw the graphs of the functions defined by the following limits, and determine their continuity.

$$(1) f(x) = \lim_{n \rightarrow \infty} \frac{x^{n+1} + x^{-n-1}}{x^n + x^{-n}}$$

[Sol]

(a) When $0 < |x| < 1$,

$$f(x) = \lim_{n \rightarrow \infty} \frac{x^{2n+1} + x^{-1}}{x^{2n} + 1} = \frac{1}{x}$$

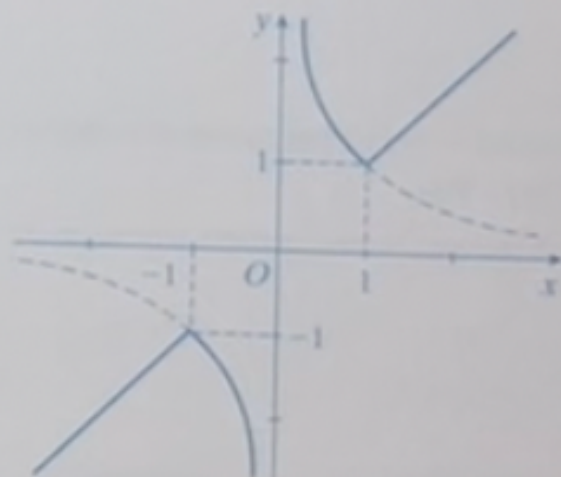
(b) When $|x| > 1$,

$$f(x) = \lim_{n \rightarrow \infty} \frac{x + \frac{1}{x^{2n+1}}}{1 + \frac{1}{x^{2n}}} = x$$

(c) When $x = 1$, $f(x) = 1$

(d) When $x = -1$, $f(x) = -1$

(e) When $x = 0$, $f(x)$ does not exist.



Therefore, the graph of $y = f(x)$ is discontinuous at $x = 0$ and continuous everywhere else.

N 164 b

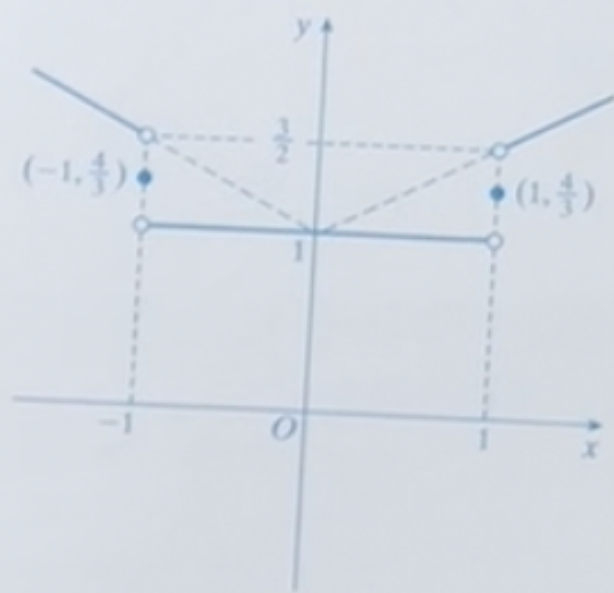
$$(2) f(x) = \lim_{n \rightarrow \infty} \frac{|x|^{2n+1} + 2x^{2n} + 1}{2x^{2n} + 1}$$

[Sol]

(a) When $|x| < 1$, $f(x) = 1$

(b) When $|x| > 1$, $f(x) = \lim_{n \rightarrow \infty} \frac{|x| + 2 + \frac{1}{x^{2n}}}{2 + \frac{1}{x^{2n}}} = \frac{|x| + 2}{2}$

(c) When $|x| = 1$, $f(x) = \frac{4}{3}$



Therefore, the graph of $y = f(x)$ is discontinuous at $x = \pm 1$ and continuous everywhere else.

Continuous and Discontinuous Functions

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	—	—	—	—

Draw the graphs of the functions defined by the following limits, and determine their continuity.

$$(1) f(x) = \lim_{n \rightarrow \infty} \frac{1}{\cos^2 \pi x + n \sin^2 \pi x}$$

[Sol]

(a) When x is an integer,

$$\sin^2 \pi x = \boxed{0}, \quad \cos^2 \pi x = \boxed{1}$$

$$\therefore f(x) = \boxed{1}$$

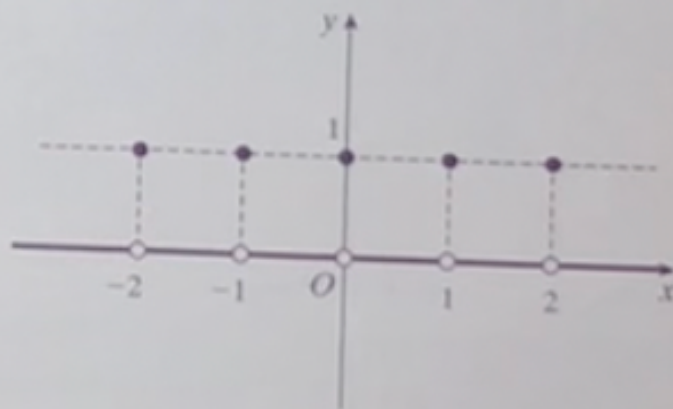
(b) When x is not an integer,

For any given x , $\cos^2 \pi x$ will be either 0 or positive.

However, $\sin^2 \pi x$ will be positive.

$$\lim_{n \rightarrow \infty} (n \sin^2 \pi x) = +\infty$$

$$\therefore f(x) = \boxed{0}$$



Therefore, the graph of $y = f(x)$ is discontinuous at $x = n$ (where n is an integer) and continuous everywhere else.

N 165 b

$$(2) f(x) = \lim_{n \rightarrow \infty} (x + \sin^{2n} x)$$

(Hint) Analyze the cases when $x \neq m\pi + \frac{\pi}{2}$ and $x = m\pi + \frac{\pi}{2}$, where m is an integer.

[Sol]

(a) When $x \neq m\pi + \frac{\pi}{2}$,

Since $|\sin x| < 1$, $0 \leq \sin^2 x < 1$

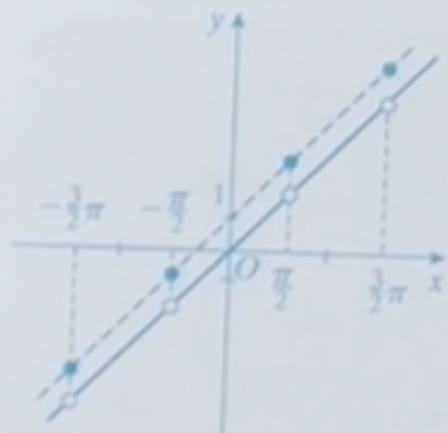
Thus, as $n \rightarrow \infty$, $\sin^{2n} x = (\sin^2 x)^n \rightarrow 0$

Therefore, $f(x) = \lim_{n \rightarrow \infty} (x + \sin^{2n} x) = x$

(b) When $x = m\pi + \frac{\pi}{2}$,

Since $|\sin x| = 1$, $\sin^{2n} x = 1$

Thus, $f(x) = \lim_{n \rightarrow \infty} (x + \sin^{2n} x) = x + 1$



Therefore, the graph of $y = f(x)$ is discontinuous at $x = m\pi + \frac{\pi}{2}$ (where m is an integer) and continuous everywhere else.

Continuous and Discontinuous Functions

Time : to : Date Name

100%	90%	80%	70%	60% -
(mistakes) 0	-	-	-	1-

Draw the graphs of the functions defined by the following limits, and determine their continuity.

$$(1) f(x) = \lim_{n \rightarrow \infty} \frac{1 - x^n + x^{n+1}}{1 - x^n + x^{n+2}}$$

[Sol]

(a) When $|x| > 1$,

$$f(x) = \lim_{n \rightarrow \infty} \frac{\frac{1}{x^n} - 1 + x}{\frac{1}{x^n} - 1 + x^2} = \frac{x - 1}{x^2 - 1} = \frac{1}{x + 1}$$

(b) When $|x| < 1$,

$$f(x) = \lim_{n \rightarrow \infty} \frac{1 - x^n + x^{n+1}}{1 - x^n + x^{n+2}} = \frac{1 - 0 + 0}{1 - 0 + 0} = 1$$

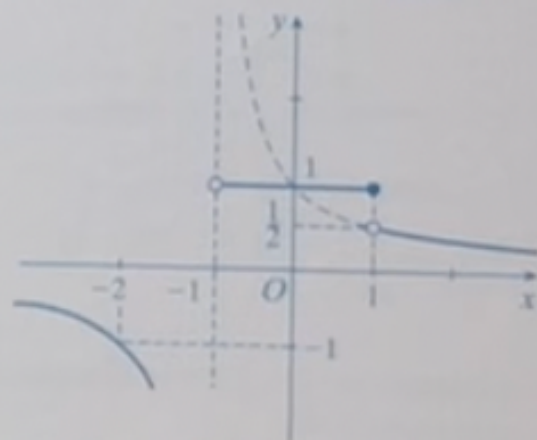
(c) When $x = 1$,

$$f(x) = \frac{1 - 1 + 1}{1 - 1 + 1} = 1$$

(d) When $x = -1$,

$$f(x) = \lim_{n \rightarrow \infty} \frac{1 - (-1)^n + (-1)^{n+1}}{1 - (-1)^n + (-1)^{n+2}}$$

The limit value changes depending on whether n is odd or even. Therefore, the limit does not exist.



Thus, the graph of $y = f(x)$ is discontinuous at $x = \pm 1$ and continuous everywhere else.

N 166 b

$$(2) f(x) = \lim_{n \rightarrow \infty} \frac{3 + [\log_2(1 + |x|)]^n}{1 + [\log_2(1 + |x|)]^n}$$

[Sol]

(a) When $|x| > 1$,

Since $1 + |x| > 2$,

$$\log_2(1 + |x|) > 1$$

$$f(x) = \lim_{n \rightarrow \infty} \frac{\frac{3}{[\log_2(1 + |x|)]^n} + 1}{\frac{1}{[\log_2(1 + |x|)]^n} + 1} = 1$$

(b) When $|x| < 1$,

Since $1 + |x| < 2$,

$$\log_2(1 + |x|) < 1$$

Also, based on the properties of log,

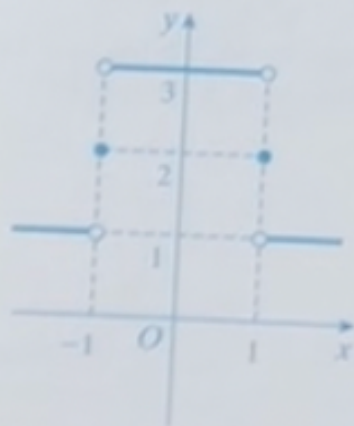
$$\log_2(1 + |x|) \geq 0$$

$$\therefore 0 \leq \log_2(1 + |x|) < 1$$

$$f(x) = \lim_{n \rightarrow \infty} \frac{3 + [\log_2(1 + |x|)]^n}{1 + [\log_2(1 + |x|)]^n} = 3$$

(c) When $|x| = 1$,

$$f(x) = \frac{3 + 1}{1 + 1} = 2$$



Therefore, the graph of $y = f(x)$ is discontinuous at $x = \pm 1$ and continuous everywhere else.

Continuous and Discontinuous Functions

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	1	2	3	4

1. In each exercise, obtain the value of the constant n at which the given function $f(x)$ is continuous for all x .

$$(1) f(x) = \begin{cases} -nx^2 - 4n - 1 & (\text{if } x < -2) \\ -nx - 9 & (\text{if } x \geq -2) \end{cases}$$

[Sol] If $f(x)$ is continuous for all x , then $f(x)$ must be continuous at $x = -2$.

Thus, $-nx^2 - 4n - 1 = -nx - 9$ when $x = -2$

$$\therefore -n(-2)^2 - 4n - 1 = -n(-2) - 9$$

$$\text{Therefore, } n = \frac{4}{5}$$

$$(2) f(x) = \begin{cases} -n|x+2| - 3 & (\text{if } x < -3) \\ nx^3 - x^2 + 2 & (\text{if } x \geq -3) \end{cases}$$

[Sol] If $f(x)$ is continuous for all x , then $f(x)$ must be continuous at $x = -3$.

Thus, $-n|x+2| - 3 = nx^3 - x^2 + 2$ when $x = -3$

$$\therefore -n|-1| - 3 = -27n - 9 + 2$$

$$\text{Therefore, } n = -\frac{2}{13}$$

N 167 b

2. Obtain the values of the constants a and b at which the following function $f(x)$ is continuous for all x .

$$f(x) = \lim_{n \rightarrow \infty} \frac{x^{2n-1} + ax^2 + bx}{x^{2n} + 1}$$

[Sol] (a) When $|x| < 1$, $f(x) = ax^2 + bx$

(b) When $|x| > 1$, $f(x) = \frac{1}{x}$

(c) When $x = 1$, $f(x) = f(1) = \frac{1 + a + b}{2}$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (ax^2 + bx) = a + b$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \frac{1}{x} = 1$$

(d) When $x = -1$, $f(x) = f(-1) = \frac{-1 + a - b}{2}$

$$\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^-} \frac{1}{x} = -1$$

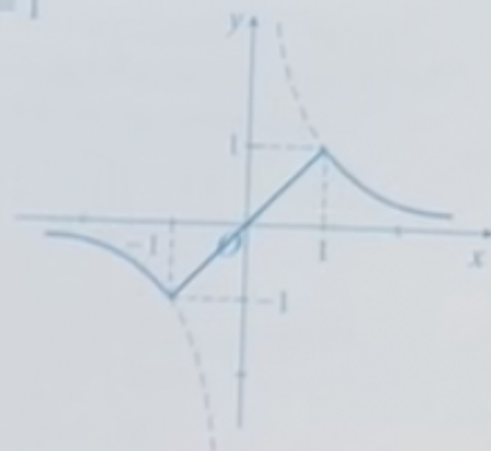
$$\lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} (ax^2 + bx) = a - b$$

Therefore, we need to find the values for $x = \pm 1$ in order to make sure it is continuous.

$$\lim_{x \rightarrow 1^-} f(x) = f(1) = \lim_{x \rightarrow 1^+} f(x) \quad \lim_{x \rightarrow -1^-} f(x) = f(-1) = \lim_{x \rightarrow -1^+} f(x)$$

$$a + b = \frac{1 + a + b}{2} = 1 \dots \textcircled{1} \quad a - b = \frac{-1 + a - b}{2} = -1 \dots \textcircled{2}$$

Solving $\textcircled{1}$ and $\textcircled{2}$, $a = 0$ and $b = 1$



Thus, when $a = 0$ and $b = 1$, the graph of $y = f(x)$ is continuous everywhere.

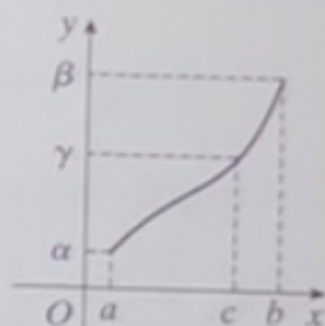
Continuous and Discontinuous Functions

Time : to : Date Name

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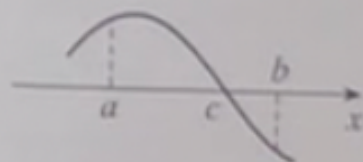
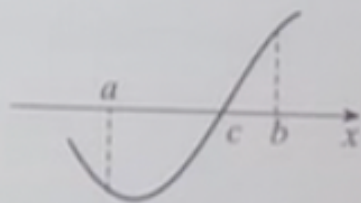
Intermediate Value Theorem

Given that a function $f(x)$ is continuous over the interval $[a, b]$, if $f(a) = \alpha$, $f(b) = \beta$, and if γ lies between α and β , then there is at least one value c , where $a < c < b$, at which $f(c) = \gamma$.



From the Intermediate Value Theorem,

Given that $y = f(x)$ is continuous over the interval $[a, b]$, where $f(a) \cdot f(b) < 0$, there is at least one value c , where $a < c < b$, at which $f(c) = 0$.



N 168 b

1. Prove that the equation $x^3 + x^2 - 2x - 5 = 0$ has at least one real number solution in the interval $[0, 2]$.

[Sol] Letting $f(x) = x^3 + x^2 - 2x - 5$, $f(x)$ is continuous over $[0, 2]$.

$$\text{Also, } f(0) = \boxed{-5} < 0, \quad f(2) = \boxed{3} > 0.$$

Therefore, from the Intermediate Value Theorem, there is at least one value c , where $\boxed{0} < c < \boxed{2}$, at which $f(c) = 0$.

2. Show that $\log_{10}x + x - 2 = 0$ has at least one real number solution between 1 and 2.

[Sol] Letting $f(x) = \log_{10}x + x - 2$, $f(x)$ is continuous over $[1, 2]$.

$$f(1) = 0 + 1 - 2 = -1 < 0$$

$$f(2) = \log_{10}2 + 2 - 2 = \log_{10}2 > 0$$

Therefore, from the Intermediate Value Theorem, there is at least one value c , where $1 < c < 2$, at which $f(c) = 0$.

3. Show that $x^4 + x - 40 = 0$ has at least one real number solution between 2 and 3.

[Sol] Letting $f(x) = x^4 + x - 40$, $f(x)$ is continuous over $[2, 3]$.

$$f(2) = -22 < 0,$$

$$f(3) = 44 > 0$$

Therefore, from the Intermediate Value Theorem, there is at least one value c , where $2 < c < 3$, at which $f(c) = 0$.

Continuous and Discontinuous Functions

Time : to : Date Name

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(mistakes) 0	-	1	-	2-

1. Show that $\sin x = x \cos x$ has at least one real number solution in the interval $\left[\pi, \frac{3}{2}\pi\right]$.

[Sol] Letting $f(x) = \sin x - x \cos x$, $f(x)$ is continuous over $\left[\pi, \frac{3}{2}\pi\right]$.

$$f(\pi) = \pi > 0 \quad \text{and} \quad f\left(\frac{3}{2}\pi\right) = -1 < 0$$

Therefore, from the Intermediate Value Theorem, there is at least one value c , where $\pi < c < \frac{3}{2}\pi$, at which $f(c) = 0$.

2. Show that $\cos x = \sin x$ has at least one real number solution in the interval $[\pi, 2\pi]$.

[Sol] Letting $f(x) = \cos x - \sin x$, $f(x)$ is continuous over $[\pi, 2\pi]$.

$$f(\pi) = -1 < 0 \quad \text{and} \quad f(2\pi) = 1 > 0$$

Therefore, from the Intermediate Value Theorem, there is at least one value c , where $\pi < c < 2\pi$, at which $f(c) = 0$.

3. Show that $\tan^2 x = \cos^2 x$ has at least one real number solution in the interval $\left[0, \frac{\pi}{4}\right]$.

[Sol] Letting $f(x) = \tan^2 x - \cos^2 x$, $f(x)$ is continuous over $\left[0, \frac{\pi}{4}\right]$.

$$f(0) = -1 < 0 \quad \text{and} \quad f\left(\frac{\pi}{4}\right) = \frac{1}{2} > 0$$

Therefore, from the Intermediate Value Theorem, there is at least one value c , where $0 < c < \frac{\pi}{4}$, at which $f(c) = 0$.

N 169 b

4. Show that the cubic equation $f(x) = (x+a)^2(x-b) + x^2 = 0$ has one positive and two negative solutions when $a > 0$ and $b > 0$.

[Sol] Rearranging $f(x)$,

$$f(x) = x^3 \left[\left(1 + \frac{a}{x}\right)^2 \left(1 - \frac{b}{x}\right) + \frac{1}{x} \right]$$

Therefore, $\lim_{x \rightarrow -\infty} f(x) = \boxed{-\infty}$ and $\lim_{x \rightarrow \infty} f(x) = \boxed{+\infty}$

Also, $f(-a) = a^2 > 0$ and $f(0) = -a^2b < 0$

Since $f(x)$ is a cubic function, it is continuous for all x .

Thus, from the Intermediate Value Theorem, $f(x)$ has at least one solution in each of the intervals $(-\infty, -a)$, $(-a, 0)$ and $(0, \infty)$.

Further, since it is a cubic function and cannot have more than three solutions, there must be two negative and one positive solution.

5. Show that the equation $f(x) = x(x-3)(x+3) + 3k(x-1)(x+1) = 0$ ($k > 0$) has three different real number solutions.

[Sol] Let $f(x) = x(x-3)(x+3) + 3k(x-1)(x+1)$

$$f(x) = x(x^2 - 9) + 3k(x^2 - 1)$$

$$= x^3 \left[\left(1 - \frac{9}{x^2}\right) + 3k \left(\frac{1}{x} - \frac{1}{x^3}\right) \right]$$

Therefore, $\lim_{x \rightarrow -\infty} f(x) = -\infty$ and $\lim_{x \rightarrow \infty} f(x) = +\infty$

Also, $f(-1) = 8 > 0$ and $f(0) = -3k < 0$

Since $f(x)$ is a cubic function, it is continuous for all x .

Thus, from the Intermediate Value Theorem, $f(x)$ has at least one solution in each of the intervals $(-\infty, -1)$, $(-1, 0)$ and $(0, \infty)$.

Further, since it is a cubic function and cannot have more than three solutions, the three real number solutions must be different.

Continuous and Discontinuous Functions

Time : to : Date Name

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1. In each of the exercises, show that the given function has at least one real number solution in the given interval.

(1) $5x^3 - 4x^2 + 2x = 1$ $[0, 1]$

[Sol] Letting $f(x) = 5x^3 - 4x^2 + 2x - 1$, $f(x)$ is continuous over $[0, 1]$.

$$f(0) = -1 < 0$$

$$f(1) = 5 - 4 + 2 - 1 = 2 > 0$$

Therefore, from the Intermediate Value Theorem, there is at least one value c , where $0 < c < 1$, at which $f(c) = 0$.

(2) $\tan x = \cos x$ $\left[0, \frac{\pi}{4}\right]$

[Sol] Letting $f(x) = \tan x - \cos x$, $f(x)$ is continuous over $\left[0, \frac{\pi}{4}\right]$.

$$f(0) = -1 < 0$$

$$f\left(\frac{\pi}{4}\right) = 1 - \frac{\sqrt{2}}{2} = \frac{2 - \sqrt{2}}{2} > 0$$

Therefore, from the Intermediate Value Theorem, there is at least one value c , where $0 < c < \frac{\pi}{4}$, at which $f(c) = 0$.

2. Determine the continuity of the function $f(x) = \lim_{n \rightarrow \infty} \frac{x^{4n}}{1 + x^{4n}}$.

[Sol] (a) When $|x| < 1$, $f(x) = \lim_{n \rightarrow \infty} \frac{x^{4n}}{1 + x^{4n}} = 0$

(b) When $|x| > 1$, $f(x) = \lim_{n \rightarrow \infty} \frac{1}{\frac{1}{x^{4n}} + 1} = 1$

(c) When $x = \pm 1$, $f(x) = \frac{1}{2}$

Therefore, $f(x)$ is discontinuous at $x = \pm 1$ and continuous everywhere else.

N 170 b

3. Draw the graph of the function defined by the following limit and determine its continuity.

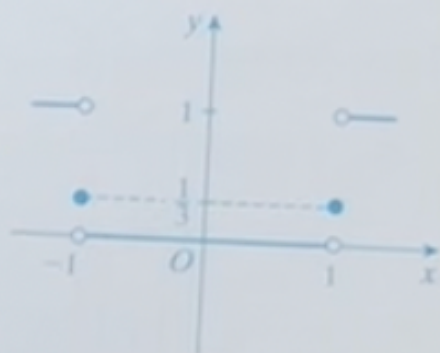
$$f(x) = \lim_{n \rightarrow \infty} \frac{x^{4n}}{x^{4n} + 2}$$

[Sol]

(a) When $|x| < 1$, $f(x) = \lim_{n \rightarrow \infty} \frac{x^{4n}}{x^{4n} + 2} = 0$

(b) When $|x| > 1$, $f(x) = \lim_{n \rightarrow \infty} \frac{x^{4n}}{x^{4n} + 2} = \frac{1}{1 + \frac{2}{x^{4n}}} = \frac{1}{1 + 0} = 1$

(c) When $x = \pm 1$, $f(x) = \frac{1}{3}$



Therefore, the graph of $y = f(x)$ is discontinuous at $x = \pm 1$ and continuous everywhere else.

Differentiation 1

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	-	-	1-

The derivative of $f(x)$

Given the function $f(x)$, if the value of $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ exists, then $f(x)$ is said to be *differentiable* at $x = a$. This is written as: $f'(a)$

Substituting x for $a + h$, the above expression can be rewritten as:

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

We have already learned that the derivative of $f(x)$ can be written as:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Letting $y = f(x)$, the derivative of $f(x)$ can also be written as:

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

(Δx denotes the change in x , Δy denotes the change in y .)

Using the above formulas, obtain the derivative value of each of the following functions at $x = 1$.

(1) $y = \frac{1}{x^3}$

[Sol] Letting $y = f(x)$,

$$\begin{aligned} f'(1) &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{(1+h)^3} - \frac{1}{1^3}}{h} \\ &= \lim_{h \rightarrow 0} \frac{-(3+3h+h^2)}{(1+h)^3} = -3 \end{aligned}$$

N 171 b

(2) $y = x^3 - x$

[Sol] Letting $y = f(x)$,

$$\begin{aligned}f'(1) &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} \\&= \lim_{h \rightarrow 0} \frac{[(1+h)^3 - (1+h)] - (1^3 - 1)}{h} \\&= \lim_{h \rightarrow 0} (h^2 + 3h + 2) \\&= 2\end{aligned}$$

(3) $y = x^2 + x + 1$

[Sol 1] Letting $y = f(x)$,

$$\begin{aligned}f'(1) &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} \\&= \lim_{h \rightarrow 0} \frac{[(1+h)^2 + (1+h) + 1] - (1^2 + 1 + 1)}{h} \\&= \lim_{h \rightarrow 0} (3 + h) \\&= 3\end{aligned}$$

[Sol 2] Letting $y = f(x)$,

$$\begin{aligned}f'(1) &= \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} \\&= \lim_{x \rightarrow 1} \frac{(x^2 + x + 1) - (1^2 + 1 + 1)}{x - 1} \\&= \lim_{x \rightarrow 1} \frac{(x-1)(x+2)}{x-1} \\&= \lim_{x \rightarrow 1} (x+2) \\&= 3\end{aligned}$$

Differentiation 1

Time : to : Date Name

100%	90%	80%	70%	60%~
(mistakes) 0	-	1	-	2-

In each of the following exercises, obtain the derivative of the given function.

(1) $y = \frac{1}{x^2}$

[Sol] Letting $y = f(x)$,

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{(x+h)^2} - \frac{1}{x^2} \right] \\&= \lim_{h \rightarrow 0} \frac{-(2x+h)}{(x+h)^2 \cdot x^2} \\&= -\frac{2}{x^3}\end{aligned}$$

(2) $y = \frac{1}{x+1}$

$$\begin{aligned}\text{[Sol]} \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{(x+h)+1} - \frac{1}{x+1} \right] \\&= \lim_{h \rightarrow 0} \frac{x+1 - [(x+h)+1]}{h(x+h+1)(x+1)} \\&= \lim_{h \rightarrow 0} \frac{-1}{(x+h+1)(x+1)} = -\frac{1}{(x+1)^2}\end{aligned}$$

(3) $y = \frac{x}{x-1}$

$$\begin{aligned}\text{[Sol]} \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{x+h}{(x+h)-1} - \frac{x}{x-1} \right] \\&= \lim_{h \rightarrow 0} \frac{x^2 - x + hx - h - x^2 - hx + x}{h(x+h-1)(x-1)} \\&= \lim_{h \rightarrow 0} \frac{-1}{(x+h-1)(x-1)} = -\frac{1}{(x-1)^2}\end{aligned}$$

N 172 b

(4) $y = \sqrt{x}$

$$\begin{aligned} [\text{Sol}] \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

(5) $y = 2\sqrt{x}$

$$\begin{aligned} [\text{Sol}] \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{2(\sqrt{x+h} - \sqrt{x})}{h} \\ &= \lim_{h \rightarrow 0} \frac{2(x+h-x)}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \rightarrow 0} \frac{2}{\sqrt{x+h} + \sqrt{x}} \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

(6) $y = \frac{1}{\sqrt{x}}$

$$\begin{aligned} [\text{Sol}] \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}} \right) \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x+h} \cdot \sqrt{x}} \\ &= \lim_{h \rightarrow 0} \frac{-1}{\sqrt{x+h} \cdot \sqrt{x}(\sqrt{x} + \sqrt{x+h})} \\ &= -\frac{1}{2x\sqrt{x}} \end{aligned}$$

Differentiation 1

Time : to : Date Name

100%	90%	80%	70%	60%
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Reviewing the Rules of Differentiation from Level L,

$$\begin{aligned}
 c' &= 0, & (\text{where } c \text{ is a constant}) \\
 (x^n)' &= nx^{n-1}, & (\text{where } n \text{ is a positive integer}) \\
 \text{Given that } u &= f(x) \text{ and } v = g(x), \\
 (cu)' &= cu', & (\text{where } c \text{ is a constant}) \\
 (u + v)' &= u' + v' \\
 (u - v)' &= u' - v'
 \end{aligned}$$

Obtain the derivative of each of the following functions.

(1) $y = x^7$

[Sol] $y' = 7x^6$

(2) $y = 2x^3 + 5x$

[Sol] $y' = 6x^2 + 5$

(3) $y = x^2 - 2x - 1$

[Sol] $y' = 2x - 2$

N 173 b

$$(4) \ y = \frac{2}{3}x^3 - x^2 + 2x - 3$$

$$[\text{Sol}] \ y' = 2x^2 - 2x + 2$$

$$(5) \ y = 2(3x - 2)$$

$$[\text{Sol}] \ y = 6x - \boxed{4}$$

$$y' = \boxed{6}$$

$$(6) \ y = x(x + 1)$$

$$[\text{Sol}] \ y = x^2 + x$$

$$y' = 2x + 1$$

$$(7) \ y = x^2(x + 1)$$

$$[\text{Sol}] \ y = x^3 + x^2$$

$$y' = 3x^2 + 2x$$

$$(8) \ y = x^3(x^2 + 2x + 1)$$

$$[\text{Sol}] \ y = x^5 + 2x^4 + x^3$$

$$y' = 5x^4 + 8x^3 + 3x^2$$

Differentiation 1

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	1	2	3	4

Product Rule

Given that u and v are functions of x , and letting the derivatives of u and v be u' and v' , respectively,

$$(uv)' = u'v + uv'$$

Using the Product Rule, obtain the derivative of each of the following functions.

(1) $y = (x + 1)(x^2 + 2)$

$$\begin{aligned} \text{[Sol]} \quad y' &= 1 \cdot (x^2 + 2) + (x + 1) \cdot 2x \\ &= 3x^2 + 2x + 2 \end{aligned}$$

(2) $y = (3x^2 + 1)(2x - 3)$

$$\begin{aligned} \text{[Sol]} \quad y' &= 6x(2x - 3) + (3x^2 + 1) \cdot 2 \\ &= 12x^2 - 18x + 6x^2 + 2 \\ &= 18x^2 - 18x + 2 \end{aligned}$$

(3) $y = (2x + 4)(-x - 3)$

$$\begin{aligned} \text{[Sol]} \quad y' &= 2 \cdot (-x - 3) + (2x + 4) \cdot (-1) \\ &= -2x - 6 - 2x - 4 \\ &= -4x - 10 \end{aligned}$$

(4) $y = (x^3 + 2x)(x^2 - 5)$

$$\begin{aligned} \text{[Sol]} \quad y' &= (3x^2 + 2)(x^2 - 5) + (x^3 + 2x) \cdot 2x \\ &= 3x^4 - 15x^2 + 2x^2 - 10 + 2x^4 + 4x^2 \\ &= 5x^4 - 9x^2 - 10 \end{aligned}$$

N 174 b

$$(5) \ y = 2x(3x - 1)(x - 1)$$

$$[\text{Sol}] \ y = (6x^2 - 2x)(x - 1)$$

$$\begin{aligned} y' &= (12x - 2)(x - 1) + (6x^2 - 2x) \cdot 1 \\ &= 12x^2 - 14x + 2 + 6x^2 - 2x \\ &= 18x^2 - 16x + 2 \end{aligned}$$

$$(6) \ y = (x - 4)(x + 3)(x + 2)$$

$$[\text{Sol}] \ y = (x^2 - x - 12)(x + 2)$$

$$\begin{aligned} y' &= (2x - 1)(x + 2) + (x^2 - x - 12) \cdot 1 \\ &= 2x^2 + 3x - 2 + x^2 - x - 12 \\ &= 3x^2 + 2x - 14 \end{aligned}$$

$$(7) \ y = (x - 2)(x + 1)(2x + 1)$$

$$[\text{Sol}] \ y = (x^2 - x - 2)(2x + 1)$$

$$\begin{aligned} y' &= (2x - 1)(2x + 1) + (x^2 - x - 2) \cdot 2 \\ &= 4x^2 - 1 + 2x^2 - 2x - 4 \\ &= 6x^2 - 2x - 5 \end{aligned}$$

$$(8) \ y = (x - 3)(x + 1)^2(x + 3)$$

$$[\text{Sol}] \ y = (x^2 - 9)(x^2 + 2x + 1)$$

$$\begin{aligned} y' &= 2x(x^2 + 2x + 1) + (x^2 - 9)(2x + 2) \\ &= 2x^3 + 4x^2 + 2x + 2x^3 + 2x^2 - 18x - 18 \\ &= 4x^3 + 6x^2 - 16x - 18 \end{aligned}$$

Differentiation 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	-	1	-	2-

Quotient
RuleGiven that u and v are functions of x , and letting the derivatives of u and v be u' and v' , respectively,

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

1. Prove the Quotient Rule by applying the following steps.

(1) Prove: $\left(\frac{1}{v}\right)' = -\frac{v'}{v^2}$

[Sol] Letting $y = F(x) = \frac{1}{f(x)}$,

$$\begin{aligned} y' &= \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{f(x+h)} - \frac{1}{f(x)} \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[-\frac{f(x+h) - f(x)}{f(x+h) \cdot f(x)} \right] \\ &= \lim_{h \rightarrow 0} \left[-\frac{1}{f(x+h) \cdot f(x)} \cdot \frac{f(x+h) - f(x)}{h} \right] \\ &= -\frac{1}{[f(x)]^2} \cdot f'(x) \end{aligned}$$

Letting $v = f(x)$,

$$\left(\frac{1}{v}\right)' = -\frac{v'}{v^2}$$

N 175 b

(2) Using $\left(\frac{1}{v}\right)' = -\frac{v'}{v^2}$, obtain $\left(\frac{u}{v}\right)'$.

$$\begin{aligned} \text{[Sol]} \left(\frac{u}{v}\right)' &= \left(u \cdot \frac{1}{v}\right)' \\ &= u' \cdot \frac{1}{v} + u \cdot \left(\frac{1}{v}\right)' \\ &= u' \cdot \frac{1}{v} - u \cdot \frac{v'}{v^2} \\ &= \frac{u'v - uv'}{v^2} \end{aligned}$$

2. Using the Quotient Rule, obtain the derivative of each of the following functions.

(1) $y = \frac{1}{3x}$

$$\text{[Sol]} y' = \frac{-3}{(3x)^2} = -\frac{1}{3x^2}$$

(2) $y = \frac{1}{x^2 + 1}$

$$\text{[Sol]} y' = -\frac{2x}{(x^2 + 1)^2}$$

(3) $y = \frac{x-1}{x+1}$

$$\text{[Sol]} y' = \frac{(x+1) - (x-1)}{(x+1)^2} = \frac{2}{(x+1)^2}$$

Differentiation 1

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	1	2	3	4

Using the Quotient Rule, obtain the derivative of each of the following functions.

$$(1) y = \frac{x+1}{3x-1}$$

$$[\text{Sol}] y' = \frac{3x-1-(x+1) \cdot 3}{(3x-1)^2} = -\frac{4}{(3x-1)^2}$$

$$(2) y = \frac{x}{1+x^2}$$

$$[\text{Sol}] y' = \frac{1+x^2-2x^2}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2} = \frac{(1+x)(1-x)}{(1+x^2)^2}$$

$$(3) y = \frac{x^2}{1-x}$$

$$[\text{Sol}] y' = \frac{2x(1-x)+x^2}{(1-x)^2} = \frac{2x-x^2}{(1-x)^2} = \frac{x(2-x)}{(1-x)^2}$$

$$(4) y = \frac{x^2-1}{x^2+1}$$

$$[\text{Sol}] y' = \frac{2x(x^2+1)-(x^2-1) \cdot 2x}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}$$

N 176 b

$$(5) \ y = \frac{x-1}{x}$$

$$[\text{Sol}] \ y' = \frac{x - (x-1)}{x^2} = \frac{1}{x^2}$$

$$(6) \ y = \frac{3x^2 - 5}{3x - 2}$$

$$\begin{aligned} [\text{Sol}] \ y' &= \frac{6x(3x-2) - (3x^2-5) \cdot 3}{(3x-2)^2} = \frac{9x^2 - 12x + 15}{(3x-2)^2} \\ &= \frac{3(3x^2 - 4x + 5)}{(3x-2)^2} \end{aligned}$$

$$(7) \ y = \frac{2x^2 + 5x}{x^3}$$

$$\begin{aligned} [\text{Sol}] \ y' &= \frac{(4x+5) \cdot x^3 - (2x^2+5x) \cdot 3x^2}{(x^3)^2} = \frac{-2x^4 - 10x^3}{x^6} \\ &= -\frac{2(x+5)}{x^3} \end{aligned}$$

Differentiation 1

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	1	2	3	4

Using the Quotient Rule, obtain the derivative of each of the following functions.

$$(1) y = \frac{2x+3}{x^2+1}$$

$$\begin{aligned} [\text{Sol}] y' &= \frac{2(x^2+1) - (2x+3) \cdot 2x}{(x^2+1)^2} \\ &= \frac{2x^2+2-4x^2-6x}{(x^2+1)^2} = -\frac{2(x^2+3x-1)}{(x^2+1)^2} \end{aligned}$$

$$(2) y = \frac{x+1}{x^2+x+1}$$

$$\begin{aligned} [\text{Sol}] y' &= \frac{x^2+x+1 - (x+1)(2x+1)}{(x^2+x+1)^2} \\ &= \frac{-x^2-2x}{(x^2+x+1)^2} = -\frac{x(x+2)}{(x^2+x+1)^2} \end{aligned}$$

$$(3) y = \frac{x^2-x+1}{x^2+x+1}$$

$$\begin{aligned} [\text{Sol}] y' &= \frac{(2x-1)(x^2+x+1) - (x^2-x+1)(2x+1)}{(x^2+x+1)^2} \\ &= \frac{2(x^2-1)}{(x^2+x+1)^2} = \frac{2(x+1)(x-1)}{(x^2+x+1)^2} \end{aligned}$$

$$(4) y = \frac{2x^2-3x+5}{x^2+2}$$

$$\begin{aligned} [\text{Sol}] y' &= \frac{(4x-3)(x^2+2) - (2x^2-3x+5) \cdot 2x}{(x^2+2)^2} \\ &= \frac{3x^2-2x-6}{(x^2+2)^2} \end{aligned}$$

N 177 b

$$(5) \ y = \frac{3x^3 + 2x + 4}{x^4 - 1}$$

$$\begin{aligned} [\text{Sol}] \ y' &= \frac{(9x^2 + 2)(x^4 - 1) - (3x^3 + 2x + 4) \cdot 4x^3}{(x^4 - 1)^2} \\ &= \frac{9x^6 + 2x^4 - 9x^2 - 2 - 12x^6 - 8x^4 - 16x^3}{(x^4 - 1)^2} \\ &= -\frac{3x^6 + 6x^4 + 16x^3 + 9x^2 + 2}{(x^4 - 1)^2} \\ &\left[= -\frac{3x^6 + 6x^4 + 16x^3 + 9x^2 + 2}{(x^2 + 1)^2(x + 1)^2(x - 1)^2} \right] \end{aligned}$$

$$(6) \ y = \frac{(x^2 + 1)(x - 1)}{x^3}$$

$$\begin{aligned} [\text{Sol}] \ [(x^2 + 1)(x - 1)]' &= 2x \cdot (x - 1) + (x^2 + 1) \cdot 1 \\ &= 2x^2 - 2x + x^2 + 1 = 3x^2 - 2x + 1 \\ \therefore y' &= \frac{(3x^2 - 2x + 1) \cdot x^3 - (x^2 + 1)(x - 1) \cdot 3x^2}{(x^3)^2} \\ &= \frac{x^4 - 2x^3 + 3x^2}{x^6} = \frac{x^2 - 2x + 3}{x^4} \end{aligned}$$

$$(7) \ y = \frac{(x - 2)(3x + 1)}{x(2x + 1)}$$

$$\begin{aligned} [\text{Sol}] \ [(x - 2)(3x + 1)]' &= 1 \cdot (3x + 1) + (x - 2) \cdot 3 = 6x - 5 \\ [x(2x + 1)]' &= 4x + 1 \\ \therefore y' &= \frac{(6x - 5) \cdot x(2x + 1) - (x - 2)(3x + 1)(4x + 1)}{x^2(2x + 1)^2} \\ &= \frac{13x^2 + 8x + 2}{x^2(2x + 1)^2} \end{aligned}$$

Differentiation 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	-	1	-	2-

Taking the derivative $f'(x)$ of the function $y = f(x)$ and differentiating it one step further, $[f'(x)]'$ gives the *second order derivative* of $f(x)$ and can be written as:

$$y'', f''(x), \frac{d^2y}{dx^2}, \text{ or } \frac{d^2}{dx^2}f(x)$$

Furthermore, differentiating the second order derivative $f''(x)$ gives the *third order derivative*, written as:

$$y''', f'''(x), \frac{d^3y}{dx^3}, \text{ or } \frac{d^3}{dx^3}f(x)$$

In general, the function obtained by differentiating the function $y = f(x)$ n times is called the n^{th} *order derivative* of $f(x)$ and can be written as:

$$y^{(n)}, f^{(n)}(x), \frac{d^ny}{dx^n}, \text{ or } \frac{d^n}{dx^n}f(x)$$

1. Obtain the second and third order derivatives of each of the following functions.

(1) $y = x^4 + 2x^3 - x + 1$

[Sol] $y' = 4x^3 + 6x^2 - 1$

$$y'' = 12x^2 + 12x$$

$$y''' = 24x + 12$$

(2) $y = \frac{1}{2}x^3 - x^2 + 4x$

[Sol] $y' = \frac{3}{2}x^2 - 2x + 4$

$$y'' = 3x - 2$$

$$y''' = 3$$

N 178 b

2. Obtain the second order derivative of each of the following functions.

$$(1) \ y = \frac{2}{x-3}$$

$$[\text{Sol}] \ y' = -\frac{2}{(x-3)^2} = -\frac{2}{x^2-6x+9}$$

$$y'' = \frac{2(2x-6)}{(x^2-6x+9)^2}$$

$$= \frac{4}{(x-3)^3}$$

$$(2) \ y = \frac{x}{x+1}$$

$$[\text{Sol}] \ y' = \frac{(x+1) - x}{(x+1)^2} = \frac{1}{x^2+2x+1}$$

$$y'' = -\frac{2x+2}{(x^2+2x+1)^2} = -\frac{2(x+1)}{(x+1)^4} = -\frac{2}{(x+1)^3}$$

$$(3) \ y = \frac{x^2+1}{x}$$

$$[\text{Sol}] \ y' = \frac{2x(x) - (x^2+1)}{x^2} = \frac{x^2-1}{x^2}$$

$$y'' = \frac{2x(x^2) - (x^2-1)(2x)}{x^4} = \frac{2}{x^3}$$

Differentiation 1

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	1	-	2-

Obtain the second order derivative of each of the following functions.

(1) $y = (x^2 + 1)(x - 3)$

[Sol] $y' = 2x(x - 3) + (x^2 + 1) = 3x^2 - 6x + 1$

$$y'' = 6x - 6$$

$$= 6(x - 1)$$

(2) $y = \frac{3x}{x - 1}$

[Sol] $y' = \frac{3(x - 1) - 3x}{(x - 1)^2} = -\frac{3}{x^2 - 2x + 1}$

$$y'' = \frac{3(2x - 2)}{(x^2 - 2x + 1)^2}$$

$$= \frac{6}{(x - 1)^3}$$

(3) $y = \frac{x + 1}{x - 1}$

[Sol] $y' = \frac{x - 1 - (x + 1)}{(x - 1)^2} = -\frac{2}{x^2 - 2x + 1}$

$$y'' = \frac{2(2x - 2)}{(x^2 - 2x + 1)^2}$$

$$= \frac{4}{(x - 1)^3}$$

N 179 b

(4) $y = x^2(2x + 3)$

[Sol] $y' = 2x(2x + 3) + x^2 \cdot 2$

$$= 4x^2 + 6x + 2x^2$$

$$= 6x^2 + 6x$$

$$y'' = 12x + 6$$

$$= 6(2x + 1)$$

(5) $y = \frac{x^2 - 2x + 1}{x + 2}$

$$[\text{Sol}] \quad y' = \frac{(2x - 2)(x + 2) - (x^2 - 2x + 1)}{(x + 2)^2} = \frac{2x^2 + 2x - 4 - x^2 + 2x - 1}{(x + 2)^2}$$

$$= \frac{x^2 + 4x - 5}{x^2 + 4x + 4}$$

$$y'' = \frac{(2x + 4)(x^2 + 4x + 4) - (x^2 + 4x - 5) \cdot 2(x + 2)}{(x + 2)^4}$$

$$= \frac{18(x + 2)}{(x + 2)^4}$$

$$= \frac{18}{(x + 2)^3}$$

Differentiation 1

Time : to : Date Name

100%	90%	80%	70%	69%
(mistakes) 0	-	1	-	2-

1. Obtain the second order derivative of each of the following functions.

(1) $y = 4x^3 + \frac{1}{2}x^2 + x$

$$\begin{aligned} \text{[Sol]} \quad y' &= 12x^2 + x + 1 \\ y'' &= 24x + 1 \end{aligned}$$

(2) $y = \frac{x-1}{x+2}$

$$\begin{aligned} \text{[Sol]} \quad y' &= \frac{(x+2) - (x-1)}{(x+2)^2} = \frac{3}{x^2 + 4x + 4} \\ y'' &= -\frac{3(2x+4)}{(x^2 + 4x + 4)^2} = -\frac{6}{(x+2)^3} \end{aligned}$$

(3) $y = \frac{x^2}{1-x}$

$$\begin{aligned} \text{[Sol]} \quad y' &= \frac{2x(1-x) - x^2(-1)}{(1-x)^2} = \frac{2x - x^2}{(1-x)^2} = -\frac{x(x-2)}{(x-1)^2} \\ y'' &= \frac{(2-2x)(1-x)^2 - (2x-x^2)(2x-2)}{(x-1)^4} \\ &= \frac{-2x+2}{(x-1)^4} \\ &= \frac{-2(x-1)}{(x-1)^4} = -\frac{2}{(x-1)^3} \left[= \frac{2}{(1-x)^3} \right] \end{aligned}$$

N 180 b

2. Obtain the first order derivative of each of the following functions.

$$(1) \ y = \frac{2x+1}{x^2+3x-1}$$

$$\begin{aligned} [\text{Sol}] \ y' &= \frac{2(x^2+3x-1) - (2x+1)(2x+3)}{(x^2+3x-1)^2} \\ &= \frac{2x^2+6x-2 - (4x^2+8x+3)}{(x^2+3x-1)^2} \\ &= -\frac{2x^2+2x+5}{(x^2+3x-1)^2} \end{aligned}$$

$$(2) \ y = \frac{x^2+1}{x^2-2}$$

$$\begin{aligned} [\text{Sol}] \ y' &= \frac{2x(x^2-2) - (x^2+1) \cdot 2x}{(x^2-2)^2} \\ &= \frac{2x^3-4x-2x^3-2x}{(x^2-2)^2} \\ &= -\frac{6x}{(x^2-2)^2} \end{aligned}$$

$$(3) \ y = 3(x+3)^2(x-1)$$

$$\begin{aligned} [\text{Sol}] \ y &= 3(x^2+6x+9)(x-1) \\ y' &= 3[(2x+6)(x-1) + (x^2+6x+9)] \\ &= 3(2x^2+4x-6+x^2+6x+9) \\ &= 3(3x^2+10x+3) \\ &= 3(3x+1)(x+3) \end{aligned}$$

Differentiation 2

Time : to : Date Name

100%	90%	80%	70%	60%~
(mistakes) 0	-	1	-	2-

- When n is a positive integer, $(x^n)' = nx^{n-1}$... ①
- When n is a negative integer,

If $n = -m$, then m is a positive integer.

$$(x^n)' = \left(\frac{1}{x^m}\right)' = \frac{-mx^{m-1}}{x^{2m}} = -mx^{-m-1} = nx^{n-1}$$

Thus, ① holds true when n is either positive or negative.

Answer: $\frac{mx^{m-1}}{x^{2m}}$

- Using the above rule, obtain the first and second order derivatives of each of the following functions.

Ex. $y = x^{-3}$

[Sol] $y' = -3x^{-4} = -\frac{3}{x^4}$

$$y'' = 12x^{-5} = \frac{12}{x^5}$$

(1) $y = \frac{1}{3}x^{-3}$

[Sol] $y' = \frac{1}{3} \cdot (-3)x^{-4} = -\frac{1}{x^4}$

$$y'' = -1 \cdot (-4)x^{-5} = \frac{4}{x^5}$$

(2) $y = 5x^{-4}$

[Sol] $y' = 5 \cdot (-4)x^{-5} = -\frac{20}{x^5}$

$$y'' = 100x^{-6} = \frac{100}{x^6}$$

N 181 b

2. Obtain the first order derivative of each of the following functions.

$$(1) \ y = \frac{(3x-1)(2x+3)}{x^3}$$

$$[\text{Sol}] \ y = \frac{6x^2 + 7x - 3}{x^3} = \frac{6}{x} + \frac{7}{x^2} - \frac{3}{x^3} = 6x^{-1} + 7x^{-2} - 3x^{-3}$$

$$\begin{aligned} y' &= 6 \cdot (-1)x^{-2} + 7 \cdot (-2)x^{-3} - 3 \cdot (-3)x^{-4} \\ &= -\frac{6}{x^2} - \frac{14}{x^3} + \frac{9}{x^4} \end{aligned}$$

$$(2) \ y = 2x^2 - \frac{3}{x} + \frac{5}{x^2}$$

$$[\text{Sol}] \ y = 2x^2 - 3x^{-1} + 5x^{-2}$$

$$\begin{aligned} y' &= 4x + 3x^{-2} - 10x^{-3} \\ &= 4x + \frac{3}{x^2} - \frac{10}{x^3} \end{aligned}$$

$$(3) \ y = \frac{2x^2 - x + 5}{3x^3}$$

$$[\text{Sol}] \ y = \frac{2}{3x} - \frac{1}{3x^2} + \frac{5}{3x^3} = \frac{2}{3}x^{-1} - \frac{1}{3}x^{-2} + \frac{5}{3}x^{-3}$$

$$\begin{aligned} y' &= -\frac{2}{3}x^{-2} + \frac{2}{3}x^{-3} - 5x^{-4} \\ &= -\frac{2}{3x^2} + \frac{2}{3x^3} - \frac{5}{x^4} \end{aligned}$$

Differentiation 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	-	1	-	2-

Differentiation Formula
for Composite Functions

$$\text{When } u = f(x) \text{ and } y = u^n,$$
$$y' = nu^{n-1} \cdot u'$$

Obtain the first order derivative of each of the following functions.

Ex.

$$y = (x^2 - 1)^3$$

$$\begin{aligned} \text{[Sol]} \quad y' &= 3(x^2 - 1)^2 \cdot (x^2 - 1)' \\ &= 3(x^2 - 1)^2 \cdot 2x \\ &= 6x(x^2 - 1)^2 \end{aligned}$$

$$(1) \quad y = -(3x - 2)^2$$

$$\begin{aligned} \text{[Sol]} \quad y' &= -2(3x - 2) \cdot 3 \\ &= -6(3x - 2) \end{aligned}$$

$$(2) \quad y = (3x^2 - 5)^{-4}$$

$$\begin{aligned} \text{[Sol]} \quad y' &= -4(3x^2 - 5)^{-5} \cdot 6x \\ &= -\frac{24x}{(3x^2 - 5)^5} \end{aligned}$$

N 182 b

$$(3) \ y = (2x^2 + 5)^6$$

$$\begin{aligned} [\text{Sol}] \ y' &= 6(2x^2 + 5)^5 \cdot 4x \\ &= 24x(2x^2 + 5)^5 \end{aligned}$$

$$(4) \ y = (x^2 + 2x + 3)^{-3}$$

$$\begin{aligned} [\text{Sol}] \ y' &= -3(x^2 + 2x + 3)^{-4} \cdot (2x + 2) \\ &= -\frac{6(x + 1)}{(x^2 + 2x + 3)^4} \end{aligned}$$

$$(5) \ y = -(x^2 + 3x + 1)^{-3}$$

$$\begin{aligned} [\text{Sol}] \ y' &= 3(x^2 + 3x + 1)^{-4} \cdot (2x + 3) \\ &= \frac{3(2x + 3)}{(x^2 + 3x + 1)^4} \end{aligned}$$

$$(6) \ y = (x^2 - 3x + 5)^4$$

$$\begin{aligned} [\text{Sol}] \ y' &= 4(x^2 - 3x + 5)^3 \cdot (2x - 3) \\ &= 4(2x - 3)(x^2 - 3x + 5)^3 \end{aligned}$$

Differentiation 2

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	1	-	2-

Obtain the first and second order derivatives of each of the following functions.

(1) $y = (-x^3 + x + 5)^2$

$$\begin{aligned}
 \text{[Sol]} \quad y' &= 2(-x^3 + x + 5) \cdot (-3x^2 + 1) \\
 &= 2(3x^2 - 1)(x^3 - x - 5) \\
 y'' &= 2[6x(x^3 - x - 5) + (3x^2 - 1)(3x^2 - 1)] \\
 &= 2(15x^4 - 12x^2 - 30x + 1)
 \end{aligned}$$

(2) $y = (1 - x^2)^3$

$$\begin{aligned}
 \text{[Sol]} \quad y' &= 3(1 - x^2)^2 \cdot (-2x) \\
 &= -6x(1 - x^2)^2 = -6x(1 + x)^2(1 - x)^2 \\
 y'' &= -6[1 \cdot (1 - x^2)^2 + x \cdot 2(1 - x^2) \cdot (-2x)] \\
 &= -6[(1 - x^2)^2 - 4x^2(1 - x^2)] \\
 &= -6(1 + x)(1 - x)(1 - 5x^2)
 \end{aligned}$$

(3) $y = (1 - 2x)^{-5}$

$$\begin{aligned}
 \text{[Sol]} \quad y' &= -5(1 - 2x)^{-6} \cdot (-2) \\
 &= 10(1 - 2x)^{-6} = \frac{10}{(1 - 2x)^6} \\
 y'' &= 10[-6(1 - 2x)^{-7} \cdot (-2)] \\
 &= 120(1 - 2x)^{-7} = \frac{120}{(1 - 2x)^7}
 \end{aligned}$$

N 183 b

(4) $y = (1 - 2x^3)^{10}$

[Sol] $y' = 10(1 - 2x^3)^9 \cdot (-6x^2)$

$$= -60x^2(1 - 2x^3)^9$$

$$y'' = -60[2x(1 - 2x^3)^9 + x^2 \cdot 9(1 - 2x^3)^8(-6x^2)]$$

$$= -60[2x(1 - 2x^3)^9 - 54x^4(1 - 2x^3)^8]$$

$$= -120x(1 - 2x^3)^8(1 - 29x^3)$$

(5) $y = (x + 1)(2x - 1)^2$

[Sol] $y' = 1 \cdot (2x - 1)^2 + (x + 1) \cdot [2(2x - 1) \cdot 2]$

$$= (2x - 1)^2 + 4(x + 1)(2x - 1)$$

$$= (2x - 1)(2x - 1 + 4x + 4)$$

$$= (2x - 1)(6x + 3) = 3(2x - 1)(2x + 1)$$

$$y'' = 3[2(2x + 1) + (2x - 1) \cdot 2]$$

$$= 24x$$

Differentiation 2

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	—	1	—	2

Obtain the first order derivative of each of the following functions.

$$(1) \quad y = \frac{1}{x-9} + \frac{3}{x+1}$$

$$\begin{aligned} \text{[Sol]} \quad y &= (x-9)^{-1} + 3(x+1)^{-1} \\ y' &= -(x-9)^{-2} - 3(x+1)^{-2} \\ &= -\frac{1}{(x-9)^2} - \frac{3}{(x+1)^2} \end{aligned}$$

$$(2) \quad y = \frac{3x^2+3}{x-3}$$

$$\begin{aligned} \text{[Sol]} \quad y &= (3x^2+3)(x-3)^{-1} \\ y' &= 6x(x-3)^{-1} + (3x^2+3) \cdot [-(x-3)^{-2}] \\ &= \frac{6x}{x-3} - \frac{3(x^2+1)}{(x-3)^2} \\ &= \frac{6x(x-3) - 3(x^2+1)}{(x-3)^2} = \frac{3(x^2-6x-1)}{(x-3)^2} \end{aligned}$$

$$(3) \quad y = \frac{x^2-2x+1}{9-x}$$

$$\begin{aligned} \text{[Sol]} \quad y &= (x^2-2x+1)(9-x)^{-1} \\ y' &= (2x-2)(9-x)^{-1} + (x^2-2x+1) \cdot [-(9-x)^{-2}] \cdot (-1) \\ &= \frac{2(x-1)}{9-x} + \frac{(x-1)^2}{(9-x)^2} \\ &= \frac{-x^2+18x-17}{(9-x)^2} = -\frac{(x-17)(x-1)}{(x-9)^2} \end{aligned}$$

N 184 b

$$(4) \ y = \frac{x^3 - 2x + 3}{x + 1}$$

$$[\text{Sol}] \ y = (x^3 - 2x + 3)(x + 1)^{-1}$$

$$\begin{aligned} y' &= (3x^2 - 2)(x + 1)^{-1} + (x^3 - 2x + 3) \cdot [-(x + 1)^{-2}] \\ &= \frac{3x^2 - 2}{x + 1} - \frac{x^3 - 2x + 3}{(x + 1)^2} \\ &= \frac{3x^3 + 3x^2 - 2x - 2 - x^3 + 2x - 3}{(x + 1)^2} \\ &= \frac{2x^3 + 3x^2 - 5}{(x + 1)^2} \quad \left[= \frac{(x - 1)(2x^2 + 5x + 5)}{(x + 1)^2} \right] \end{aligned}$$

$$(5) \ y = \frac{1}{(x - 1)(x - 2)}$$

$$[\text{Sol}] \ y = (x - 1)^{-1}(x - 2)^{-1}$$

$$\begin{aligned} y' &= -(x - 1)^{-2}(x - 2)^{-1} + (x - 1)^{-1} \cdot [-(x - 2)^{-2}] \\ &= -\frac{1}{(x - 1)^2(x - 2)} - \frac{1}{(x - 1)(x - 2)^2} \\ &= -\frac{2x - 3}{(x - 1)^2(x - 2)^2} \end{aligned}$$

$$(6) \ y = \frac{3}{2(x - 4)(x + 1)}$$

$$[\text{Sol}] \ y = \frac{3}{2}(x - 4)^{-1}(x + 1)^{-1}$$

$$\begin{aligned} y' &= \frac{3}{2} \{ -(x - 4)^{-2}(x + 1)^{-1} + (x - 4)^{-1} \cdot [-(x + 1)^{-2}] \} \\ &= \frac{3}{2} \left[-\frac{1}{(x - 4)^2(x + 1)} - \frac{1}{(x - 4)(x + 1)^2} \right] \\ &= -\frac{3}{2} \cdot \frac{(x + 1) + (x - 4)}{(x - 4)^2(x + 1)^2} = -\frac{3(2x - 3)}{2(x - 4)^2(x + 1)^2} \end{aligned}$$

Differentiation 2

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	1	-	2-

Obtain the first order derivative of each of the following functions.

(1) $y = (1 - x + 2x^2 - x^3)^{-4}$

$$\begin{aligned} \text{[Sol]} \quad y' &= -4(1 - x + 2x^2 - x^3)^{-5}(-1 + 4x - 3x^2) \\ &= \frac{4(3x - 1)(x - 1)}{(1 - x + 2x^2 - x^3)^5} \end{aligned}$$

(2) $y = \left(\frac{1}{3}x + 2x^3\right)^5$

$$\begin{aligned} \text{[Sol]} \quad y' &= 5\left(\frac{1}{3}x + 2x^3\right)^4\left(\frac{1}{3} + 6x^2\right) \\ &= \frac{5}{243}x^4(1 + 6x^2)^4(1 + 18x^2) \end{aligned}$$

(3) $y = (x + 2x^2)^3$

$$\begin{aligned} \text{[Sol]} \quad y' &= 3(x + 2x^2)^2 \cdot (1 + 4x) \\ &= 3x^2(1 + 2x)^2(1 + 4x) \end{aligned}$$

N 185 b

(4) $y = (3x^2 - 2)^{-3}$

[Sol] $y' = -3(3x^2 - 2)^{-4} \cdot 6x$

$$= -\frac{18x}{(3x^2 - 2)^4}$$

(5) $y = (2x^3 - 6x + 1)^{-3}$

[Sol] $y' = -3(2x^3 - 6x + 1)^{-4} \cdot (6x^2 - 6)$

$$= -3(2x^3 - 6x + 1)^{-4} \cdot 6(x - 1)(x + 1)$$

$$= -\frac{18(x - 1)(x + 1)}{(2x^3 - 6x + 1)^4}$$

(6) $y = \frac{1}{2x - 3} + \frac{x}{x + 5}$

[Sol] $y = (2x - 3)^{-1} + x(x + 5)^{-1}$

$$y' = -(2x - 3)^{-2} \cdot 2 + \{1 \cdot (x + 5)^{-1} + x \cdot [-(x + 5)^{-2}]\}$$

$$= -\frac{2}{(2x - 3)^2} + \left[\frac{1}{x + 5} - \frac{x}{(x + 5)^2} \right]$$

$$= -\frac{2}{(2x - 3)^2} + \frac{5}{(x + 5)^2}$$

Differentiation 2

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	—	—	1	2—

1. Obtain the first order derivative of the function $y = \left(\frac{x}{x+1}\right)^3$.

[Sol] Using the formula on N182a,

$$\begin{aligned}y' &= 3 \cdot \left(\frac{x}{x+1}\right)^2 \cdot \left(\frac{x}{x+1}\right)' \\&= \boxed{\frac{3x^2}{(x+1)^4}}\end{aligned}$$

Note: This method is easier than having to first expand the function and then use the Quotient Rule.

2. Obtain the first order derivative of each of the following functions.

(1) $y = \left(\frac{2x-1}{x^3+7}\right)^{-2}$

$$\begin{aligned}[\text{Sol}] \quad y' &= -2 \left(\frac{2x-1}{x^3+7}\right)^{-3} \cdot \left(\frac{2x-1}{x^3+7}\right)' \\&= -2 \left(\frac{2x-1}{x^3+7}\right)^{-3} \cdot \frac{2(x^3+7) - (2x-1)(3x^2)}{(x^3+7)^2} \\&= -2 \left(\frac{x^3+7}{2x-1}\right)^3 \cdot \frac{-4x^3+3x^2+14}{(x^3+7)^2} \\&= \frac{2(x^3+7)(4x^3-3x^2-14)}{(2x-1)^3}\end{aligned}$$

N 186 b

$$(2) \ y = \left(\frac{3x}{1-2x+x^2} \right)^{-4}$$

$$\begin{aligned} [\text{Sol}] \ y' &= -4 \left(\frac{3x}{1-2x+x^2} \right)^{-5} \cdot \frac{3(1-2x+x^2) - 3x(-2+2x)}{(1-2x+x^2)^2} \\ &= -4 \left(\frac{1-2x+x^2}{3x} \right)^5 \cdot \frac{-3x^2+3}{(1-2x+x^2)^2} \\ &= -\frac{4(1-x)^2(x+1)}{81x^5} \end{aligned}$$

$$(3) \ y = \left(\frac{-x^2+x-5}{2x-3} \right)^2$$

$$\begin{aligned} [\text{Sol}] \ y' &= 2 \left(\frac{-x^2+x-5}{2x-3} \right) \cdot \left(\frac{-x^2+x-5}{2x-3} \right)' \\ &= 2 \left(\frac{-x^2+x-5}{2x-3} \right) \cdot \frac{(-2x+1)(2x-3) - (-x^2+x-5) \cdot 2}{(2x-3)^2} \\ &= \frac{2(x^2-x+5)(2x^2-6x-7)}{(2x-3)^3} \end{aligned}$$

Differentiation 2

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	-	1	2-

1. Obtain the first and second order derivatives of each of the following functions.

(1) $y = \frac{1}{(x-1)^2}$

[Sol] $y' = 2 \cdot \frac{1}{x-1} \cdot \left(\frac{1}{x-1}\right)'$

$$= 2 \cdot \frac{1}{x-1} \cdot \frac{-1}{(x-1)^2}$$

$$= -\frac{2}{(x-1)^3}$$

$$y'' = 6(x-1)^{-4}$$

$$= \frac{6}{(x-1)^4}$$

(2) $y = \left(\frac{x}{x-1}\right)^3$

[Sol] $y' = 3\left(\frac{x}{x-1}\right)^2 \left(\frac{x}{x-1}\right)'$

$$= 3\left(\frac{x}{x-1}\right)^2 \left[-\frac{1}{(x-1)^2}\right]$$

$$= -\frac{3x^2}{(x-1)^4}$$

$$y'' = -3[2x(x-1)^{-4} + x^2 \cdot (-4)(x-1)^{-5}]$$

$$= -3[2x(x-1)^{-5}(x-1-2x)]$$

$$= \frac{6x(x+1)}{(x-1)^5}$$

N 187 b

2. Obtain the first order derivative of each of the following functions.

$$(1) \ y = \left(x - \frac{1}{x}\right)^3$$

$$\begin{aligned} [\text{Sol}] \ y' &= 3\left(x - \frac{1}{x}\right)^2 \left(x - \frac{1}{x}\right)' \\ &= 3\left(x - \frac{1}{x}\right)^2 \left(1 + \frac{1}{x^2}\right) \\ &= \frac{3(x^2 - 1)^2(x^2 + 1)}{x^4} \\ &= \frac{3(x^2 + 1)(x + 1)^2(x - 1)^2}{x^4} \end{aligned}$$

$$(2) \ y = \left(\frac{x+1}{x^2+2}\right)^3$$

$$\begin{aligned} [\text{Sol}] \ y' &= 3\left(\frac{x+1}{x^2+2}\right)^2 \left(\frac{x+1}{x^2+2}\right)' \\ &= 3\left(\frac{x+1}{x^2+2}\right)^2 \cdot \frac{x^2+2 - (x+1) \cdot 2x}{(x^2+2)^2} \\ &= \frac{-3(x+1)^2(x^2+2x-2)}{(x^2+2)^4} \end{aligned}$$

Differentiation 2

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	-	1	2-

1. Obtain the first and second order derivatives of each of the following functions.

(1) $y = \frac{x^3}{(1-x)^5}$

[Sol] $y' = \frac{3x^2(1-x)^5 - x^3 \cdot 5(1-x)^4 \cdot (-1)}{(1-x)^{10}}$

$= \frac{(1-x)^4(3x^2 - 5x^3 + 5x^3)}{(1-x)^{10}}$

$= \frac{2x^3 + 3x^2}{(1-x)^6} = \frac{x^2(2x+3)}{(1-x)^6}$

$y'' = \frac{(6x^2 + 6x)(1-x)^6 - (2x^3 + 3x^2) \cdot 6(1-x)^5(-1)}{(1-x)^{12}}$

$= \frac{6x(x+1)(1-x)^6 + 6x^2(2x+3)(1-x)^5}{(1-x)^{12}}$

$= \frac{6x(x^2 + 3x + 1)}{(1-x)^7}$

(2) $y = \frac{2a(x^2-1)}{(x+1)^2}$ (where a is a constant)

[Sol] $y = \frac{2a(x-1)}{x+1}$

$y' = \frac{2a(x+1) - 2a(x-1)}{(x+1)^2}$

$= \frac{4a}{(x+1)^2}$

$y'' = \frac{0 \cdot (x+1)^2 - 4a \cdot 2(x+1)}{(x+1)^4}$

$= -\frac{8a}{(x+1)^3}$

N 188 b

2. Obtain the first order derivative of each of the following functions.

$$(1) y = \left(\frac{5x}{2x-3} \right)^2$$

$$\begin{aligned} [\text{Sol}] \quad y' &= 2 \left(\frac{5x}{2x-3} \right) \cdot \left(\frac{5x}{2x-3} \right)' \\ &= 2 \left(\frac{5x}{2x-3} \right) \cdot \left[-\frac{15}{(2x-3)^2} \right] \\ &= -\frac{150x}{(2x-3)^3} \end{aligned}$$

$$(2) y = \left(\frac{2x+3}{x^2-1} \right)^3$$

$$\begin{aligned} [\text{Sol}] \quad y' &= 3 \left(\frac{2x+3}{x^2-1} \right)^2 \left(\frac{2x+3}{x^2-1} \right)' \\ &= 3 \left(\frac{2x+3}{x^2-1} \right)^2 \cdot \frac{2(x^2-1) - (2x+3) \cdot 2x}{(x^2-1)^2} \\ &= 3 \left(\frac{2x+3}{x^2-1} \right)^2 \cdot \frac{-2x^2-6x-2}{(x^2-1)^2} \\ &= -\frac{6(2x+3)^2(x^2+3x+1)}{(x-1)^4(x+1)^4} \end{aligned}$$

Differentiation 2

Time : to : Date Name

100%	90%	80%	70%	60%~
(mistakes) 0	-	-	1	2-

1. Obtain the first and second order derivatives of each of the following functions.

(1) $y = \left(\frac{1}{2x+1}\right)^2$

$$\begin{aligned}[\text{Sol}] \quad y' &= 2\left(\frac{1}{2x+1}\right)\left(\frac{1}{2x+1}\right)' \\&= \frac{2}{2x+1} \cdot \left[-\frac{2}{(2x+1)^2}\right] \\&= -\frac{4}{(2x+1)^3}\end{aligned}$$

$$\begin{aligned}y'' &= 12(2x+1)^{-4} \cdot (2) \\&= \frac{24}{(2x+1)^4}\end{aligned}$$

(2) $y = \frac{(x-1)(x-2)}{(x+3)^3}$

$$\begin{aligned}[\text{Sol}] \quad y' &= \frac{(2x-3)(x+3)^3 - (x-1)(x-2) \cdot 3(x+3)^2}{(x+3)^6} \\&= \frac{(x+3)^2[(2x-3)(x+3) - 3(x-1)(x-2)]}{(x+3)^6} \\&= \frac{-x^2 + 12x - 15}{(x+3)^4}\end{aligned}$$

$$\begin{aligned}y'' &= \frac{(-2x+12)(x+3)^4 - (-x^2+12x-15) \cdot 4(x+3)^3}{(x+3)^8} \\&= \frac{2(x+3)^3[(-x+6)(x+3) + 2(x^2-12x+15)]}{(x+3)^8} \\&= \frac{2[(-x^2+3x+18) + (2x^2-24x+30)]}{(x+3)^5} \\&= \frac{2(x^2-21x+48)}{(x+3)^5}\end{aligned}$$

N 189 b

2. Obtain the first order derivative of each of the following functions.

$$(1) \ y = \left(\frac{x^2 + 12}{x + 2} \right)^4$$

$$\begin{aligned} [\text{Sol}] \ y' &= 4 \left(\frac{x^2 + 12}{x + 2} \right)^3 \cdot \left(\frac{x^2 + 12}{x + 2} \right)' \\ &= 4 \left(\frac{x^2 + 12}{x + 2} \right)^3 \cdot \frac{2x(x + 2) - (x^2 + 12)}{(x + 2)^2} \\ &= \frac{4(x^2 + 12)^3(x + 6)(x - 2)}{(x + 2)^5} \end{aligned}$$

$$(2) \ y = \left(\frac{x + 1}{x^2 + 2} \right)^3$$

$$\begin{aligned} [\text{Sol}] \ y' &= 3 \left(\frac{x + 1}{x^2 + 2} \right)^2 \left(\frac{x + 1}{x^2 + 2} \right)' \\ &= 3 \left(\frac{x + 1}{x^2 + 2} \right)^2 \cdot \frac{x^2 + 2 - (x + 1) \cdot 2x}{(x^2 + 2)^2} \\ &= \frac{-3(x + 1)^2(x^2 + 2x - 2)}{(x^2 + 2)^4} \end{aligned}$$

Differentiation 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	-	-	1	2-

1. Obtain the first and second order derivatives of the function $y = \frac{x^2 + 1}{3x^2 - 1}$.

$$\begin{aligned} \text{[Sol]} \quad y' &= \frac{2x(3x^2 - 1) - (x^2 + 1) \cdot 6x}{(3x^2 - 1)^2} \\ &= -\frac{8x}{(3x^2 - 1)^2} \\ y'' &= -\frac{8(3x^2 - 1)^2 - 8x \cdot 2(3x^2 - 1) \cdot 6x}{(3x^2 - 1)^4} \\ &= \frac{8(3x^2 - 1)(9x^2 + 1)}{(3x^2 - 1)^4} \\ &= \frac{8(9x^2 + 1)}{(3x^2 - 1)^3} \end{aligned}$$

2. Obtain the first order derivative of each of the following functions.

(1) $y = (x - 2)(3x + 1)^2$

$$\begin{aligned} \text{[Sol]} \quad y' &= (x - 2)'(3x + 1)^2 + (x - 2)[(3x + 1)^2]' \\ &= (3x + 1)^2 + (x - 2)[2(3x + 1) \cdot 3] \\ &= (3x + 1)^2 + 6(x - 2)(3x + 1) \\ &= (3x + 1)(3x + 1 + 6x - 12) \\ &= (3x + 1)(9x - 11) \end{aligned}$$

N 190 b

$$(2) \ y = \left(x - \frac{1}{x^2}\right)^2$$

$$[\text{Sol}] \ y = (x - x^{-2})^2$$

$$y' = 2(x - x^{-2}) \cdot (x - x^{-2})'$$

$$= 2\left(x - \frac{1}{x^2}\right) \cdot (1 + 2x^{-3})$$

$$= 2\left(x - \frac{1}{x^2}\right)\left(1 + \frac{2}{x^3}\right)$$

$$\left[= \frac{2(x^6 + x^3 - 2)}{x^5} \right]$$

$$(3) \ y = \frac{3}{x-2} + \frac{x}{2x-1}$$

$$[\text{Sol}] \ y = 3(x-2)^{-1} + x(2x-1)^{-1}$$

$$y' = -3(x-2)^{-2} + \{1 \cdot (2x-1)^{-1} + x \cdot [-(2x-1)^{-2}] \cdot 2\}$$

$$= -\frac{3}{(x-2)^2} + \frac{1}{2x-1} - \frac{2x}{(2x-1)^2}$$

$$= -\frac{3}{(x-2)^2} - \frac{1}{(2x-1)^2}$$

$$\left[= -\frac{13x^2 - 16x + 7}{(x-2)^2(2x-1)^2} \right]$$

Differentiation 3

Time : to : Date Name

100%	90%	80%	70%	60%~
(mistakes) 0	-	-	1	2-

Assume that $f^{-1}(x)$ is the inverse function of a function $f(x)$.

Letting $y = f^{-1}(x)$,

$$x = f(y) \quad \dots \textcircled{1}$$

Differentiating both sides of $\textcircled{1}$ with respect to x ,

$$\frac{d}{dx}x = \frac{d}{dx}f(y)$$

$$1 = \frac{d}{dy}f(y) \cdot \frac{dy}{dx} = \frac{d}{dy}x \cdot \frac{dy}{dx} \quad \therefore 1 = \frac{dx}{dy} \cdot \frac{dy}{dx}$$

Therefore, if $\frac{dx}{dy} \neq 0$, then: $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

**Rule of Differentiation
for Inverse Functions**

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

- Using the above rule, differentiate the function $y = x^{\frac{1}{m}}$, where m is a non-zero integer.

[Sol] From $y = x^{\frac{1}{m}}$, $x = y^m$

$$\frac{dx}{dy} = \boxed{my^{m-1}}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\boxed{\frac{dx}{dy}}} = \frac{1}{my^{m-1}} = \frac{1}{m(x^{\frac{1}{m}})^{m-1}} = \frac{1}{m}x^{\boxed{\frac{1}{m}-1}}$$

Note:

From the above, the formula of $(x^n)' = nx^{n-1}$ holds true even when n is the reciprocal of an integer.

N 191 b

2. Differentiate the function $y = x^{\frac{1}{p}}$, where p and q are integers and $p > 0$.

[Sol] Rearranging the equation of the function,

$$y = (x^{\frac{1}{p}})^q$$

$$y' = q(x^{\frac{1}{p}})^{q-1} \cdot (x^{\frac{1}{p}})'$$

$$= qx^{\frac{q-1}{p}} \cdot \frac{1}{p}x^{\frac{1}{p}-1}$$

$$= \frac{q}{p}x^{\frac{1}{p}-1}$$

Rule $(x^n)' = nx^{n-1}$ (where n is a rational number)

3. Obtain the first and second order derivatives of each of the following functions.

Ex.

$$y = x^{\frac{1}{2}}$$

$$[\text{Sol}] y' = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$y'' = -\frac{1}{4}x^{-\frac{3}{2}} = -\frac{1}{4x\sqrt{x}}$$

(1) $y = x^{\frac{2}{3}}$

$$[\text{Sol}] y' = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}$$

$$y'' = -\frac{2}{9}x^{-\frac{4}{3}} = -\frac{2}{9x\sqrt[3]{x}}$$

(2) $y = x^{-\frac{1}{2}}$

$$[\text{Sol}] y' = -\frac{1}{2}x^{-\frac{3}{2}} = -\frac{1}{2x\sqrt{x}}$$

$$y'' = \frac{3}{4}x^{-\frac{5}{2}} = \frac{3}{4x^2\sqrt{x}}$$

Differentiation 3

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	—	1	—	2—

Obtain the first and second order derivatives of each of the following functions.

$$(1) y = \frac{1}{\sqrt[6]{x}}$$

$$[\text{Sol}] y = x^{\boxed{-\frac{1}{6}}}$$

$$y' = -\frac{1}{6}x^{-\frac{1}{6}} = -\frac{1}{6x\sqrt[6]{x}}$$

$$y'' = \frac{7}{36}x^{-\frac{7}{6}} = \frac{7}{36x^2\sqrt[6]{x}}$$

$$(2) y = \frac{1}{\sqrt[3]{x^2}}$$

$$[\text{Sol}] y = x^{-\frac{2}{3}}$$

$$y' = -\frac{2}{3}x^{-\frac{2}{3}} = -\frac{2}{3x\sqrt[3]{x^2}}$$

$$y'' = \frac{10}{9}x^{-\frac{5}{3}} = \frac{10}{9x^2\sqrt[3]{x^2}}$$

$$(3) y = 2x^2 + \frac{1}{\sqrt[4]{x}}$$

$$[\text{Sol}] y = 2x^2 + x^{-\frac{1}{4}}$$

$$y' = 4x - \frac{1}{4}x^{-\frac{1}{4}} = 4x - \frac{1}{4x\sqrt[4]{x}}$$

$$y'' = 4 + \frac{5}{16}x^{-\frac{5}{4}} = 4 + \frac{5}{16x^2\sqrt[4]{x}}$$

N 192 b

$$(4) y = x(2\sqrt{x} - 3)$$

$$[\text{Sol}] y = 2x^{\frac{3}{2}} - 3x$$

$$y' = 2 \cdot \frac{3}{2} x^{\frac{1}{2}} - 3$$

$$= 3\sqrt{x} - 3$$

$$y'' = \frac{3}{2} x^{-\frac{1}{2}} = \frac{3}{2\sqrt{x}}$$

$$(5) y = \frac{3}{2} \sqrt[3]{x^2} - \frac{2}{\sqrt{x}}$$

$$[\text{Sol}] y = \frac{3}{2} x^{\frac{2}{3}} - 2x^{-\frac{1}{2}}$$

$$y' = \frac{3}{2} \cdot \frac{2}{3} x^{-\frac{1}{3}} - 2 \cdot \left(-\frac{1}{2}\right) x^{-\frac{3}{2}} = x^{-\frac{1}{3}} + x^{-\frac{3}{2}} = \frac{1}{\sqrt[3]{x}} + \frac{1}{x\sqrt{x}}$$

$$y'' = -\frac{1}{3} x^{-\frac{4}{3}} - \frac{3}{2} x^{-\frac{5}{2}} = -\frac{1}{3x\sqrt[3]{x}} - \frac{3}{2x^2\sqrt{x}}$$

$$(6) y = \sqrt{x\sqrt{x}}$$

$$[\text{Sol}] y = [x \cdot (x^{\frac{1}{2}})]^{\frac{1}{2}} = x^{\frac{3}{4}}$$

$$y' = \frac{3}{4} x^{-\frac{1}{4}} = \frac{3}{4\sqrt[4]{x}}$$

$$y'' = -\frac{3}{16} x^{-\frac{5}{4}} = -\frac{3}{16x\sqrt[4]{x}}$$

Differentiation 3

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	1	-	2

Obtain the first and second order derivatives of each of the following functions.

(1) $y = \sqrt[5]{x} + \frac{1}{\sqrt[3]{x}}$

[Sol] $y = x^{\frac{1}{5}} + x^{-\frac{1}{3}}$

$$y' = \frac{1}{5}x^{-\frac{4}{5}} - \frac{1}{3}x^{-\frac{4}{3}} = \frac{1}{5\sqrt[5]{x^4}} - \frac{1}{3x\sqrt[3]{x}}$$

$$y'' = -\frac{4}{25}x^{-\frac{9}{5}} + \frac{4}{9}x^{-\frac{7}{3}} = -\frac{4}{25x\sqrt[5]{x^4}} + \frac{4}{9x^2\sqrt[3]{x}}$$

(2) $y = \frac{2}{\sqrt{x}} + \sqrt[4]{x^3}$

[Sol] $y = 2x^{-\frac{1}{2}} + x^{\frac{3}{4}}$

$$y' = -x^{-\frac{3}{2}} + \frac{3}{4}x^{-\frac{1}{4}} = -\frac{1}{x\sqrt{x}} + \frac{3}{4\sqrt[4]{x}}$$

$$y'' = \frac{3}{2}x^{-\frac{5}{2}} - \frac{3}{16}x^{-\frac{5}{4}} = \frac{3}{2x^2\sqrt{x}} - \frac{3}{16x\sqrt[4]{x}}$$

(3) $y = \frac{1}{\sqrt{x-2}} + \frac{1}{\sqrt{x+1}}$

[Sol] $y = (x-2)^{-\frac{1}{2}} + (x+1)^{-\frac{1}{2}}$

$$y' = -\frac{1}{2}(x-2)^{-\frac{3}{2}} - \frac{1}{2}(x+1)^{-\frac{3}{2}}$$

$$= -\frac{1}{2(x-2)\sqrt{x-2}} - \frac{1}{2(x+1)\sqrt{x+1}}$$

$$y'' = \frac{3}{4}(x-2)^{-\frac{5}{2}} + \frac{3}{4}(x+1)^{-\frac{5}{2}}$$

$$= \frac{3}{4(x-2)^2\sqrt{x-2}} + \frac{3}{4(x+1)^2\sqrt{x+1}}$$

N 193 b

$$(4) y = \sqrt{2x+3}$$

$$[\text{Sol}] y = (2x+3)^{\frac{1}{2}}$$

$$y' = \frac{1}{2}(2x+3)^{-\frac{1}{2}} \cdot 2 = \frac{1}{\sqrt{2x+3}}$$

$$y'' = -\frac{1}{2}(2x+3)^{-\frac{3}{2}} \cdot 2 = -\frac{1}{(2x+3)\sqrt{2x+3}}$$

$$(5) y = \sqrt[3]{1-4x}$$

$$[\text{Sol}] y = (1-4x)^{\frac{1}{3}}$$

$$y' = \frac{1}{3}(1-4x)^{-\frac{2}{3}} \cdot (-4) = -\frac{4}{3\sqrt[3]{(1-4x)^2}}$$

$$y'' = \frac{8}{9}(1-4x)^{-\frac{5}{3}} \cdot (-4) = -\frac{32}{9(1-4x)\sqrt[3]{(1-4x)^2}}$$

$$(6) y = \frac{5}{\sqrt[3]{3x-4}}$$

$$[\text{Sol}] y = 5(3x-4)^{-\frac{1}{3}}$$

$$y' = -\frac{5}{3}(3x-4)^{-\frac{4}{3}} \cdot 3 = -\frac{5}{(3x-4)\sqrt[3]{3x-4}}$$

$$y'' = \frac{20}{3}(3x-4)^{-\frac{7}{3}} \cdot 3 = \frac{20}{(3x-4)^2\sqrt[3]{3x-4}}$$

Differentiation 3

Time : to : Date Name

100%	90%	80%	70%	60%~
(mistakes) ()	-	1	-	2-

Obtain the first order derivative of each of the following functions.

(1) $y = \sqrt[3]{3x+2}$

[Sol] $y = (3x+2)^{\frac{1}{3}}$

$$y' = \frac{1}{3}(3x+2)^{-\frac{2}{3}} \cdot 3$$

$$= \frac{1}{\sqrt[3]{(3x+2)^2}}$$

(2) $y = \sqrt{2x^2-3x+5}$

[Sol] $y = (2x^2-3x+5)^{\frac{1}{2}}$

$$y' = \frac{1}{2}(2x^2-3x+5)^{-\frac{1}{2}} \cdot (4x-3)$$

$$= \frac{4x-3}{2\sqrt{2x^2-3x+5}}$$

(3) $y = (x + \sqrt{x^2+1})^2$

[Sol] $y = [x + (x^2+1)^{\frac{1}{2}}]^2$

$$y' = 2[x + (x^2+1)^{\frac{1}{2}}] \cdot \left[1 + \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \cdot 2x \right]$$

$$= \frac{2(x + \sqrt{x^2+1})^2}{\sqrt{x^2+1}}$$

N 194 b

$$(4) \ y = \left(x - \frac{1}{\sqrt{x}}\right)^2$$

$$[\text{Sol}] \ y = (x - x^{-1/2})^2$$

$$\begin{aligned} y' &= 2(x - x^{-1/2}) \cdot \left(1 + \frac{1}{2}x^{-3/2}\right) \\ &= 2\left(x - \frac{1}{\sqrt{x}}\right)\left(1 + \frac{1}{2x\sqrt{x}}\right) \\ &= \frac{2x^3 - x\sqrt{x} - 1}{x^2} \end{aligned}$$

$$(5) \ y = \sqrt[3]{\frac{1}{8}x^2 + x}$$

$$[\text{Sol}] \ y = \left(\frac{1}{8}x^2 + x\right)^{1/3}$$

$$\begin{aligned} y' &= \frac{1}{3}\left(\frac{1}{8}x^2 + x\right)^{-2/3} \cdot \left(\frac{1}{4}x + 1\right) \\ &= \frac{\frac{x}{4} + 1}{3\sqrt[3]{\frac{1}{64}x^2(x+8)^2}} = \frac{\frac{x+4}{4}}{\frac{3}{4}\sqrt[3]{x^2(x+8)^2}} \\ &= \frac{x+4}{3\sqrt[3]{x^2(x+8)^2}} \end{aligned}$$

Differentiation 3

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	-	1	2-

Obtain the first and second order derivatives of each of the following functions.

(1) $y = \sqrt[3]{-3x+2}$

[Sol] $y = (-3x+2)^{\frac{1}{3}}$

$$y' = \frac{1}{3}(-3x+2)^{-\frac{2}{3}} \cdot (-3) = -\frac{1}{\sqrt[3]{(-3x+2)^2}}$$

$$y'' = \frac{2}{3}(-3x+2)^{-\frac{5}{3}} \cdot (-3) = -2(-3x+2)^{-\frac{5}{3}} = -\frac{2}{(-3x+2)\sqrt[3]{(-3x+2)^2}}$$

(2) $y = \sqrt{x^2+x+1}$

[Sol] $y = (x^2+x+1)^{\frac{1}{2}}$

$$y' = \frac{1}{2}(x^2+x+1)^{-\frac{1}{2}} \cdot (2x+1)$$

$$= \frac{2x+1}{2\sqrt{x^2+x+1}}$$

$$y'' = \frac{2(2\sqrt{x^2+x+1}) - (2x+1) \cdot 2 \cdot \frac{1}{2}[(x^2+x+1)]^{-\frac{1}{2}} \cdot (2x+1)}{4(x^2+x+1)}$$

$$= \frac{4\sqrt{x^2+x+1} - \frac{(2x+1)^2}{\sqrt{x^2+x+1}}}{4(x^2+x+1)}$$

$$= \frac{3}{4(x^2+x+1)\sqrt{x^2+x+1}}$$

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$$(3) \ y = (2\sqrt{x} + 1)^3$$

$$[\text{Sol}] \ y = (2x^{\frac{1}{2}} + 1)^3$$

$$y' = 3(2x^{\frac{1}{2}} + 1)^2 \cdot \left(2 \cdot \frac{1}{2} x^{-\frac{1}{2}}\right)$$

$$= \frac{3}{\sqrt{x}} (2\sqrt{x} + 1)^2$$

$$y'' = 3\left(-\frac{1}{2}x^{-\frac{1}{2}}\right) \cdot (2\sqrt{x} + 1)^2 + \frac{3}{\sqrt{x}} \cdot 2(2\sqrt{x} + 1)(x^{-\frac{1}{2}})$$

$$= -\frac{3(2\sqrt{x} + 1)^2}{2x\sqrt{x}} + \frac{6(2\sqrt{x} + 1)}{x}$$

$$= \frac{3(4x - 1)}{2x\sqrt{x}}$$

$$(4) \ y = \sqrt{4 - x^2}$$

$$[\text{Sol}] \ y = (4 - x^2)^{\frac{1}{2}}$$

$$y' = \frac{1}{2}(4 - x^2)^{-\frac{1}{2}} \cdot (-2x)$$

$$= -\frac{x}{\sqrt{(2-x)(2+x)}}$$

$$y'' = (-1) \cdot (4 - x^2)^{-\frac{1}{2}} + (-x) \cdot \left[-\frac{1}{2}(4 - x^2)^{-\frac{1}{2}}\right](-2x)$$

$$= -\frac{1}{\sqrt{(2-x)(2+x)}} - \frac{x^2}{(2-x)(2+x)\sqrt{(2-x)(2+x)}}$$

$$\left[= -\frac{4}{(2-x)(2+x)\sqrt{(2-x)(2+x)}} \right]$$

Differentiation 3

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	1	-	2-

Obtain the first order derivative of each of the following functions.

(1) $y = (x+4)\sqrt{x^2+4}$

[Sol] $y = (x+4)(x^2+4)^{\frac{1}{2}}$

$$\begin{aligned}y' &= 1 \cdot (x^2+4)^{\frac{1}{2}} + (x+4) \cdot \frac{1}{2}(x^2+4)^{-\frac{1}{2}}(2x) \\&= \sqrt{x^2+4} + \frac{x(x+4)}{\sqrt{x^2+4}} = \frac{2(x^2+2x+2)}{\sqrt{x^2+4}}\end{aligned}$$

(2) $y = (2x+1)\sqrt{2-x^2}$

[Sol] $y = (2x+1)(2-x^2)^{\frac{1}{2}}$

$$\begin{aligned}y' &= 2(2-x^2)^{\frac{1}{2}} + (2x+1) \cdot \frac{1}{2}(2-x^2)^{-\frac{1}{2}} \cdot (-2x) \\&= 2\sqrt{2-x^2} - \frac{x(2x+1)}{\sqrt{2-x^2}} \\&= \frac{-4x^2 - x + 4}{\sqrt{2-x^2}}\end{aligned}$$

(3) $y = (x+3)\sqrt[3]{3-2x}$

[Sol] $y = (x+3)(3-2x)^{\frac{1}{3}}$

$$\begin{aligned}y' &= (3-2x)^{\frac{1}{3}} + (x+3) \cdot \frac{1}{3}(3-2x)^{-\frac{2}{3}} \cdot (-2) \\&= \sqrt[3]{3-2x} - \frac{2(x+3)}{3\sqrt[3]{(3-2x)^2}} \\&= \frac{3-8x}{3\sqrt[3]{(3-2x)^2}}\end{aligned}$$

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$$(4) y = \frac{x}{\sqrt{x+1}}$$

$$[\text{Sol}] y = x(x+1)^{-1/2}$$

$$\begin{aligned} y' &= (x+1)^{-1/2} + x \cdot \left[-\frac{1}{2} (x+1)^{-3/2} \right] \\ &= \frac{1}{\sqrt{x+1}} - \frac{x}{2(x+1)\sqrt{x+1}} \\ &= \frac{x+2}{2(x+1)\sqrt{x+1}} \end{aligned}$$

$$(5) y = \frac{x}{\sqrt{2x^2+1}}$$

$$[\text{Sol}] y = x(2x^2+1)^{-1/2}$$

$$\begin{aligned} y' &= 1 \cdot (2x^2+1)^{-1/2} + x \cdot \left[-\frac{1}{2} (2x^2+1)^{-3/2} \right] (4x) \\ &= \frac{1}{\sqrt{2x^2+1}} - \frac{2x^2}{(2x^2+1)\sqrt{2x^2+1}} = \frac{1}{(2x^2+1)\sqrt{2x^2+1}} \end{aligned}$$

$$(6) y = \sqrt{\frac{x^2+1}{x+1}}$$

$$[\text{Sol}] y = \left(\frac{x^2+1}{x+1} \right)^{1/2}$$

$$\begin{aligned} y' &= \frac{1}{2} \left(\frac{x^2+1}{x+1} \right)^{-1/2} \left(\frac{x^2+1}{x+1} \right)' \\ &= \frac{1}{2} \cdot \sqrt{\frac{x+1}{x^2+1}} \cdot \frac{2x(x+1) - (x^2+1)}{(x+1)^2} \\ &= \frac{1}{2} \cdot \sqrt{\frac{x+1}{x^2+1}} \cdot \frac{x^2+2x-1}{(x+1)^2} = \frac{(x^2+2x-1)\sqrt{x+1}}{2(x+1)^2\sqrt{x^2+1}} \end{aligned}$$

Differentiation 3

Time : to : Date Name

100%	90%	80%	70%	60%
(mistakes) 0	-	-	1	2-

Obtain the first and second order derivatives of each of the following functions.

(1) $y = (x^2 + 4)\sqrt{2 - x^2}$

[Sol] $y = (x^2 + 4)(2 - x^2)^{\frac{1}{2}}$

$$y' = 2x(2 - x^2)^{\frac{1}{2}} + (x^2 + 4) \cdot \frac{1}{2}(2 - x^2)^{-\frac{1}{2}}(-2x)$$
$$= 2x\sqrt{2 - x^2} - \frac{x(x^2 + 4)}{\sqrt{2 - x^2}} = -\frac{3x^3}{\sqrt{2 - x^2}}$$

$$y'' = \frac{(-9x^2)\sqrt{2 - x^2} - (-3x^3) \cdot \frac{1}{2\sqrt{2 - x^2}}(-2x)}{2 - x^2}$$
$$= \frac{6x^2(x^2 - 3)}{(2 - x^2)\sqrt{2 - x^2}}$$

(2) $y = (x + 1)\sqrt{x^2 + 1}$

[Sol] $y = (x + 1)(x^2 + 1)^{\frac{1}{2}}$

$$y' = (x^2 + 1)^{\frac{1}{2}} + (x + 1) \cdot \frac{1}{2}(x^2 + 1)^{-\frac{1}{2}} \cdot 2x$$
$$= \sqrt{x^2 + 1} + \frac{x^2 + x}{\sqrt{x^2 + 1}}$$

$$= \frac{2x^2 + x + 1}{\sqrt{x^2 + 1}}$$
$$y'' = \frac{(4x + 1)\sqrt{x^2 + 1} - (2x^2 + x + 1) \cdot \frac{1}{2\sqrt{x^2 + 1}} \cdot 2x}{x^2 + 1}$$
$$= \frac{(4x + 1)(x^2 + 1) - x(2x^2 + x + 1)}{(x^2 + 1)\sqrt{x^2 + 1}} = \frac{2x^3 + 3x + 1}{(x^2 + 1)\sqrt{x^2 + 1}}$$

N 197 b

$$(3) \ y = \frac{x}{\sqrt{x^2+1}}$$

$$[\text{Sol}] \ y = x(x^2+1)^{-1/2}$$

$$y' = (x^2+1)^{-1/2} + x \cdot \left[-\frac{1}{2}(x^2+1)^{-3/2} \right] \cdot 2x$$

$$= \frac{1}{\sqrt{x^2+1}} - \frac{x^2}{(x^2+1)\sqrt{x^2+1}} = \frac{1}{(x^2+1)\sqrt{x^2+1}}$$

$$y'' = [(x^2+1)^{-3/2}]'$$

$$= -\frac{3}{2}(x^2+1)^{-5/2} \cdot (2x)$$

$$= -\frac{3x}{(x^2+1)^2\sqrt{x^2+1}}$$

$$(4) \ y = \frac{2x}{\sqrt{9-x^2}}$$

$$[\text{Sol}] \ y = 2x(9-x^2)^{-1/2}$$

$$y' = 2(9-x^2)^{-1/2} + 2x \cdot \left[-\frac{1}{2}(9-x^2)^{-3/2} \right] \cdot (-2x)$$

$$= \frac{2}{\sqrt{9-x^2}} + \frac{2x^2}{(9-x^2)\sqrt{9-x^2}} = \frac{18}{(9-x^2)\sqrt{9-x^2}}$$

$$y'' = [18(9-x^2)^{-3/2}]'$$

$$= 18 \cdot \left(-\frac{3}{2} \right) (9-x^2)^{-5/2} \cdot (-2x)$$

$$= \frac{54x}{(9-x^2)^2\sqrt{9-x^2}} \quad \left[= \frac{54x}{(3+x)^2(3-x)^2\sqrt{(3+x)(3-x)}} \right]$$

Differentiation 3

Time : to : Date Name

100%	90%	80%	70%	60%~
(mistakes) 0	-	-	1	2-

Obtain the first order derivative of each of the following functions.

(1) $y = \sqrt[3]{\frac{x^2}{1-x}}$

[Sol] $y = x^{\frac{2}{3}} \cdot (1-x)^{-\frac{1}{3}}$

$$\begin{aligned} y' &= \frac{2}{3}x^{-\frac{1}{3}} \cdot (1-x)^{-\frac{1}{3}} + x^{\frac{2}{3}} \cdot \left(-\frac{1}{3}\right)(1-x)^{-\frac{4}{3}} \cdot (-1) \\ &= \frac{2}{3}x^{-\frac{1}{3}} \cdot (1-x)^{-\frac{1}{3}} + \frac{1}{3}x^{\frac{2}{3}}(1-x)^{-\frac{4}{3}} \\ &= \frac{1}{3}x^{-\frac{1}{3}}(1-x)^{-\frac{4}{3}}[2(1-x) + x] \\ &= \frac{2-x}{3(1-x)\sqrt[3]{x(1-x)}} \end{aligned}$$

(2) $y = \sqrt[4]{\frac{x^3}{3-x^2}}$

[Sol] $y = (x^{\frac{3}{4}}) \cdot (3-x^2)^{-\frac{1}{4}}$

$$\begin{aligned} y' &= \frac{3}{4}x^{-\frac{1}{4}} \cdot (3-x^2)^{-\frac{1}{4}} + x^{\frac{3}{4}} \cdot \left[-\frac{1}{4}(3-x^2)^{-\frac{5}{4}}\right](-2x) \\ &= \frac{1}{4}x^{-\frac{1}{4}}(3-x^2)^{-\frac{5}{4}}[3(3-x^2) + 2x(x)] \\ &= -\frac{(x+3)(x-3)}{4(3-x^2)\sqrt[4]{x(3-x^2)}} \end{aligned}$$

$$(3) \ y = \sqrt[4]{\frac{x^3+1}{x^2+x-3}}$$

$$[\text{Sol}] \ y = (x^3+1)^{\frac{1}{4}} \cdot (x^2+x-3)^{-\frac{1}{4}}$$

$$\begin{aligned} y' &= \frac{1}{4}(x^3+1)^{-\frac{1}{4}}(3x^2) \cdot (x^2+x-3)^{-\frac{1}{4}} \\ &\quad + (x^3+1)^{\frac{1}{4}} \cdot \left[-\frac{1}{4}(x^2+x-3)^{-\frac{1}{4}} \right] (2x+1) \\ &= \frac{1}{4}(x^3+1)^{-\frac{1}{4}}(x^2+x-3)^{-\frac{1}{4}} [3x^2(x^2+x-3) - (x^3+1)(2x+1)] \\ &= \frac{x^4+2x^3-9x^2-2x-1}{4(x^2+x-3)\sqrt[4]{(x^3+1)^3(x^2+x-3)}} \end{aligned}$$

$$(4) \ y = \sqrt[4]{\frac{2x^2-4x+1}{x^3-5}}$$

$$[\text{Sol}] \ y = (2x^2-4x+1)^{\frac{1}{4}} \cdot (x^3-5)^{-\frac{1}{4}}$$

$$\begin{aligned} y' &= \frac{1}{4}(2x^2-4x+1)^{-\frac{1}{4}}(4x-4) \cdot (x^3-5)^{-\frac{1}{4}} \\ &\quad + (2x^2-4x+1)^{\frac{1}{4}} \cdot \left[-\frac{1}{4}(x^3-5)^{-\frac{1}{4}} \right] (3x^2) \\ &= \frac{1}{4}(2x^2-4x+1)^{-\frac{1}{4}}(x^3-5)^{-\frac{1}{4}} [(4x-4)(x^3-5) - (2x^2-4x+1)(3x^2)] \\ &= \frac{-2x^4+8x^3-3x^2-20x+20}{4(x^3-5)\sqrt[4]{(x^3-5)(2x^2-4x+1)^3}} \\ &\quad \left[= -\frac{(x-2)^2(2x^2-5)}{4(x^3-5)\sqrt[4]{(x^3-5)(2x^2-4x+1)^3}} \right] \end{aligned}$$

Differentiation 3

Time : to : Date Name:

100%	90%	80%	70%	60%
(minutes) 0				

Obtain the first order derivative of each of the following functions.

$$(1) y = \frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}}$$

$$[\text{Sol}] y = \frac{(\sqrt{1+x^2} - \sqrt{1-x^2})^2}{(\sqrt{1+x^2} + \sqrt{1-x^2})(\sqrt{1+x^2} - \sqrt{1-x^2})}$$

$$= \frac{1 - \sqrt{1-x^4}}{x^2}$$

$$y' = \frac{-\frac{1}{2\sqrt{1-x^4}}(-4x^3) \cdot x^2 - (1 - \sqrt{1-x^4}) \cdot 2x}{x^4}$$

$$= \frac{2(1 - \sqrt{1-x^4})}{x^3\sqrt{1-x^4}}$$

$$(2) y = \frac{\sqrt{x^2+1}}{(2-x)^2}$$

$$[\text{Sol}] y = (x^2+1)^{\frac{1}{2}} \cdot (2-x)^{-2}$$

$$y' = \frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x) \cdot (2-x)^{-2} + (x^2+1)^{\frac{1}{2}} \cdot [-2(2-x)^{-3}](-1)$$

$$= \frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2-x)^{-3}[2x(2-x) + 4(x^2+1)]$$

$$= \frac{x^2 + 2x + 2}{(2-x)^3\sqrt{x^2+1}}$$

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$$(3) \ y = \frac{(\sqrt{x^2 - 7})^3}{(2 - x)^2}$$

$$[\text{Sol}] \ y = (x^2 - 7)^{\frac{3}{2}} \cdot (2 - x)^{-2}$$

$$\begin{aligned} y' &= \frac{3}{2}(x^2 - 7)^{\frac{1}{2}}(2x) \cdot (2 - x)^{-2} + (x^2 - 7)^{\frac{3}{2}} \cdot [-2(2 - x)^{-3}](-1) \\ &= \frac{3}{2}(x^2 - 7)^{\frac{1}{2}}(2 - x)^{-3} \left[2x(2 - x) + \frac{4}{3}(x^2 - 7) \right] \\ &= \frac{3\sqrt{x^2 - 7}}{2(2 - x)^3} \cdot \left(-\frac{2}{3}x^2 + 4x - \frac{28}{3} \right) \\ &= -\frac{(x^2 - 6x + 14)\sqrt{x^2 - 7}}{(2 - x)^3} \end{aligned}$$

$$(4) \ y = \sqrt[8]{x \sqrt{x \sqrt{x}}}$$

$$\begin{aligned} [\text{Sol}] \ y &= [x \cdot (x \cdot x^{\frac{1}{2}})^{\frac{1}{2}}]^{\frac{1}{8}} = [x(x^{\frac{3}{2}})^{\frac{1}{2}}]^{\frac{1}{8}} \\ &= (x \cdot x^{\frac{3}{4}})^{\frac{1}{8}} = (x^{\frac{7}{4}})^{\frac{1}{8}} = x^{\frac{7}{32}} \end{aligned}$$

$$y' = \frac{7}{32}x^{-\frac{5}{32}} = \frac{7}{8\sqrt[32]{x}}$$

Differentiation 3

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	-	-	1	2-

1. Obtain the first and second order derivatives of each of the following functions.

(1) $y = \sqrt[3]{4x+1}$

[Sol] $y = (4x+1)^{\frac{1}{3}}$

$$y' = \frac{1}{3}(4x+1)^{-\frac{2}{3}}(4)$$

$$= \frac{4}{3\sqrt[3]{(4x+1)^2}}$$

$$y'' = \frac{4}{3} \cdot \left[-\frac{2}{3}(4x+1)^{-\frac{5}{3}} \right] (4) = -\frac{32}{9(4x+1)\sqrt[3]{(4x+1)^2}}$$

(2) $y = 3x^3 + \frac{2}{\sqrt[4]{x}}$

[Sol] $y = 3x^3 + 2x^{-\frac{1}{4}}$

$$y' = 9x^2 - \frac{1}{2}x^{-\frac{5}{4}} = 9x^2 - \frac{1}{2x\sqrt[4]{x}}$$

$$y'' = 18x + \frac{5}{8}x^{-\frac{9}{4}} = 18x + \frac{5}{8x^2\sqrt[4]{x}}$$

N 200 b

2. Obtain the first order derivative of each of the following functions.

$$(1) y = \frac{1}{x + \sqrt{x^2 + 1}}$$

$$[\text{Sol}] y = [x + (x^2 + 1)^{\frac{1}{2}}]^{-1}$$

$$y' = -[x + (x^2 + 1)^{\frac{1}{2}}]^{-2} \left[1 + \frac{1}{2}(x^2 + 1)^{-\frac{1}{2}} \cdot (2x) \right]$$

$$= - \frac{1 + \frac{x}{\sqrt{x^2 + 1}}}{(x + \sqrt{x^2 + 1})^2}$$

$$= - \frac{1}{(x + \sqrt{x^2 + 1})\sqrt{x^2 + 1}}$$

$$(2) y = \left(3x - \frac{1}{\sqrt{x}} \right)^2$$

$$[\text{Sol}] y = (3x - x^{-\frac{1}{2}})^2$$

$$y' = 2(3x - x^{-\frac{1}{2}}) \left(3 + \frac{1}{2}x^{-\frac{3}{2}} \right)$$

$$= 2 \left(3x - \frac{1}{\sqrt{x}} \right) \left(3 + \frac{1}{2x\sqrt{x}} \right)$$

$$= \frac{18x^3 - 3x\sqrt{x} - 1}{x^2}$$